$$M_k'' = \sup_{x \in [y, x_k]} f(x) \qquad m_k'' = \inf_{x \in [y, x_k]} f(x)$$

Then $m \le m_k \le m_k' \le M_k' \le M_k \le M$

Also $m \le m_k \le m_k'' \le M_k'' \le M_k \le M$

$$U(P,f) - U(P_1,f) = M_k(x_k - x_{k-1}) - \left[M_k'(y - x_{k-1}) + M_k''(x_k - y)\right]$$

$$= (M_k - M_k')(y - x_{k-1}) + (M_k - M_k'')(x_k - y).$$

Since $0 \le M_k - M_k' \le M - m$ and $0 \le M_k - M_k'' \le M - m$, it follows that $0 \le U(P,f) - U(P_1,f) \le (M-m)\left[(y - x_{k-1}) + (x_k - y)\right]$

$$\le (M-m)\delta, \text{ since } (x_k - x_{k-1}) \le \delta$$

Similarly, we can prove that $0 \le L(P_1,f) - L(P,f) \le (M-m)\delta$

By introducing k additional points one by one in the partition P, we obtain the partition P_k and it follows that

$$0 \le U(P,f) - U(P_k,f) \le (M-m)k\delta,$$

$$0 \le L(P_k,f) - L(P,f) \le (M-m)k\delta.$$

Lemma 1.4.2 Let a function $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and P, Q any two partitions of $[a,b]$. Then

$$L(P,f) \le U(Q,f)\,; \quad L(Q,f) \le U(P,f)$$

Proof: For any partition P, we know that $L(P,f) \le U(P,f)$.

Now let $S = P \cup Q$. Then S is a refinement of P as well as a refinement of Q

So, by Lemma 1.4.1 $L(P,f) \le L(S,f) \le U(S,f) \le U(P,f)$

and $L(Q, f) \le L(S,f) \le U(S,f) \le U(Q,f)$

So, $L(P,f) \le L(S,f) \le U(S,f) \le U(Q,f)$

and $L(Q,f) \le L(S,f) \le U(S,f) \le U(P,f)$

So, the result follows.

Note: The lemma states that for any two partition in $\mathcal{P}[a,b]$, the lower sum corresponding to one does not exceed the upper sum corresponding to the other.

Theorem 1.4.3 Let a function $f: [a,b] \to \mathbb{R}$ be bounded on $[a,b]$.

Then $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$

Proof: Let P, Q be any two partitions of $[a,b]$.

Then $L(P,f) \le U(Q,f)$. Keeping Q fixed, this inequality holds for ~~ev~~ every partition $P \in \mathcal{P}[a,b]$.

So, $U(Q,f)$ is an upper bound of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$

But the least upper bound of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$ is $\underline{\int_a^b} f(x)\,dx$. So, $\underline{\int_a^b} f(x)\,dx \le U(Q,f)$

This holds for all partition $Q \in \mathcal{P}[a,b]$. So, $\underline{\int_a^b} f(x)\,dx$ is a lower bound of the set $\{U(Q,f) : Q \in \mathcal{P}[a,b]\}$.

But the greatest lower bound is $\overline{\int_a^b} f(x)\,dx$. So,

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx.$$

Note: ~~As~~ $m(b-a) \le L(P,f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(P,f) \le M(b-a)$

So, $m(b-a) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le M(b-a)$

Theorem 1.4.4 (Darboux's theorem)

Let $[a,b]$ be a closed and bounded interval and a function $f:[a,b]\to\mathbb{R}$ be bounded on $[a,b]$. Then

(i) to every $\varepsilon>0$ $\exists$ a $\delta>0$ such that

$U(P,f) < \overline{\int_a^b} f(x)dx + \varepsilon$ for all partitions P of $[a,b]$ such that $\|P\|<\delta$;

(ii) to every $\varepsilon>0$ there $\exists$ a $\delta>0$ such that

$L(P,f) > \underline{\int_a^b} f(x)dx - \varepsilon$ for all partitions P of $[a,b]$ such that $\|P\|<\delta$

Proof: Since f is bounded on $[a,b]$, $\exists$ a positive real number B such that $|f(x)|\le B$ for all $x\in[a,b]$

Let $M = \sup\limits_{x\in[a,b]} f(x)$ and $m = \inf\limits_{x\in[a,b]} f(x)$

Then $-B\le m\le M\le B$. So, $M-m\le 2B$.

Since $\overline{\int_a^b} f(x)dx$ is the infimum of the set $\{U(P,f) : P\in\mathcal{P}[a,b]\}$, for a given $\varepsilon>0$, $\exists$ a partition Q of $[a,b]$ such that $U(Q,f) < \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2}$ ...(1)

Let $Q=(x_0,x_1,x_2,\ldots,x_n)$ and let $\delta = \frac{\varepsilon}{4B(n-1)}$.

Let P be any partition of $[a,b]$ such that $\|P\|<\delta$.

Let $P' = P\cup Q$. Then P' is a refinement of P by adjoining additional points $x_1,x_2,\ldots,x_{n-1}$ at most.

So, $0\le U(P,f)-U(P',f)\le (M-m)(n-1)\delta \le 2B(n-1)\delta$, by Lemma 1.4.1 or, $U(P,f) < U(P',f)+\frac{\varepsilon}{2}$

Since P' is also a refinement of Q,

$$U(P', f) \leq U(Q, f)$$

It follows that $U(P,f) \leq U(P',f) + \frac{\varepsilon}{2} \leq U(Q,f) + \frac{\varepsilon}{2}$

$$< \overline{\int_a^b} f(x)dx + \varepsilon$$

[from (1)]

(ii) ~~Since f(x) is bounded on [a,b],~~

Since $\underline{\int_a^b} f(x)dx$ is the supremum of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$, for given $\varepsilon > 0$ $\exists$ a partition R of $[a,b]$ such that

$$L(R,f) > \underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2}$$

Let $R = (x_0, x_1, x_2, \ldots, x_n)$ and let $\delta = \frac{\epsilon}{4B(n-1)}$

Let P be any partition of $[a,b]$ such that $\|P\| \leq \delta$

Let $P' = P \cup R$. Then P' is a refinement of P by adjoining $(n-1)$ additional points $x_1, x_2, \ldots, x_{n-1}$ at most.

So, $0 \leq L(P',f) - L(P,f) \leq (M-m)(n-1)\delta \leq 2(n-1)B\delta$,

by ~~Lemma~~ Lemma 1-4.1

or, $L(P,f) \geq L(P',f) - \frac{\varepsilon}{2}$

Since P' is also a refinement of R,

$L(P',f) \geq L(R,f)$. It follows that

$L(P,f) \geq L(R,f) - \frac{\varepsilon}{2} \geq \underline{\int_a^b} f(x)dx - \varepsilon$