1.4.5 Darboux's definition of integrability (Definition 1) :

Let $f : [a,b] \to \mathbb{R}$ be a bounded function.

When $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$

we say that f is Riemann Integrable (or simply integrable) over [a,b] and the common value is called the Riemann Integral (or simply the integral) of f over [a,b]

The class of all Riemann Integrable functions on [a,b] are denoted by $\mathcal{R}[a,b]$

Now as $m(b-a) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le M(b-a)$, so

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \quad \ldots (1)$$

1.4.6. Riemann's definition of integrability (The integral as a limit of sums (Riemann's sum)) (Definition 2) :

Earlier, we arrived at the integral of a function via the upper sum and lower sums. The numbers M_i, m_i which appear in these sums are not necessarily the values of the function f (they are values of f if f is continuous). We shall show that if $f : [a,b] \to \mathbb{R}$ be a ~~bounded~~ function, then $\int_a^b f(x)\,dx$ can be considered as ~~a~~ the limit of a sequence of sums in which M_i and m_i are replaced by the values of f.

Corresponding to a partition $P = (x_0, x_1, \ldots, x_n)$ of [a,b], let us choose points $t_1, t_2, \ldots, t_n$ such that $x_{i-1} \le t_i \le x_i$ $(i = 1, 2, \ldots, n)$ and let us consider the sum

$$S(P, f) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i \quad \text{where } \Delta x_i = x_i - x_{i-1}$$

The sum $S(P, f)$ is called a Riemann sum of f over $[a, b]$ relative to P.

It may be noted that t_i are arbitrary and t_i can be any point of $[x_{i-1}, x_{i}]$

We say that $S(P, f)$ converges to A as $\|P\| \to 0$,

i.e., $\lim\limits_{\|P\| \to 0} S(P, f) = A$ if, for every $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$|S(P, f) - A| < \varepsilon \quad \text{for every}$$

partition $P = (x_0, x_1, \ldots, x_n)$ of $[a, b]$ with $\|P\| < \delta$ and for every choice of points t_i in $[x_{i-1}, x_i]$

<u>Definition 2</u> (Riemann's definition of Integrability): A function f is said to be integrable on $[a, b]$, if $\lim S(P, f)$ exists as $\|P\| \to 0$, and then

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} S(P, f).$$

Note: Since $\|P\| \to 0$ as $n \to \infty$, therefore $\lim\limits_{\|P\| \to 0}$ can be replaced by $\lim\limits_{n \to \infty}$ in the above definition

The two definitions are equivalent, i.e.,
If a function f is integrable on $[a,b]$ in the sense of definition 2, then f is bounded in $[a,b]$ and f is integrable in the sense of definition 1, i.e., in the sense of Darboux's definition and
Conversely, if f is a bounded function in $[a,b]$ and integrable in the Darboux's sense then it is integrable in the Riemann's sense
(Only statement is in the syllabus).

1.4.7 Condition of integrability:
Let a function $f:[a,b] \to \mathbb{R}$ be bounded on $[a,b]$.
Then f is integrable on $[a,b]$ if and only if for each $\varepsilon>0$ $\exists$ a partition P of $[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon$$

<u>Proof:</u> Let $f \in \mathcal{R}[a,b]$ i.e., f be integrable on $[a,b]$.

Then $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$

Let us choose $\varepsilon>0$. Since $\underline{\int_a^b} f(x)\,dx$ is the least upper bound of the set $\{L(P,f) : P \in \mathcal{P}[a,b]\}$, $\exists$ a partition P' of $[a,b]$ such that

$$\underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2} < L(P',f) \leq \underline{\int_a^b} f(x)\,dx$$

Since $\overline{\int_a^b} f(x)\,dx$ is the greatest lower bound of the set $\{U(P,f) : P \in \mathcal{P}[a,b]\}$, $\exists$ a partition

P' of $[a,b]$ such that

$$\overline{\int_a^b} f(x)\,dx \le U(P',f) < \overline{\int_a^b} f(x)\,dx + \varepsilon/2$$

Let $P = P' \cup P''$. Then P is a refinement of both P' and P''. So,

$$L(P',f) \le L(P,f) \text{ and } U(P,f) \le U(P'',f)$$

Also, we have $L(P,f) \le U(P,f)$

Combining, we have

$$\underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2} < L(P',f) \le L(P,f) \le U(P,f) \le U(P'',f) < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2}$$

So, $U(P,f) - L(P,f) < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2} - \left(\underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2}\right)$

or, $U(P,f) - L(P,f) < \varepsilon$, since $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$

To prove the converse, we first observe that for any partition P of $[a,b]$, $L(P,f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(P,f)$

Hence $\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le U(P,f) - L(P,f)$

Let us choose $\varepsilon > 0$. By the condition, $\exists$ a partition P_ε of $[a,b]$ such that $U(P_\varepsilon,f) - L(P_\varepsilon,f) < \varepsilon$

We have $\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le U(P_\varepsilon,f) - L(P_\varepsilon,f) < \varepsilon$

Also, we have $\overline{\int_a^b} f(x)\,dx \ge \underline{\int_a^b} f(x)\,dx$, by

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So, $0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \varepsilon$. This holds for every positive ε. It follows that $\overline{\int_a^b} f(x) = \underline{\int_a^b} f(x)\,dx$ and hence f is integrable on $[a,b]$.