Theorem 1.1.8 Let a function $f:[a,b]\to\mathbb{R}$ be bounded on $[a,b]$. If $\{P_n\}$ be a sequence of partitions of $[a,b]$ such that the sequence $\{\|P_n\|\}$ converges to 0, then

(i) $\lim\limits_{n\to\infty} U(P_n, f) = \overline{\int_a^b} f(x)\,dx$ (ii) $\lim\limits_{n\to\infty} L(P_n, f) = \underline{\int_a^b} f(x)\,dx$

Proof: (i) Let us choose $\varepsilon>0$. Since f is bounded on $[a,b]$, by Darboux's theorem $\exists$ a $\delta>0$ such that

$$\overline{\int_a^b} f(x)\,dx \le U(P,f) < \overline{\int_a^b} f(x)\,dx + \varepsilon$$ for all partitions P satisfying $\|P\|<\delta$

Since $\lim\limits_{n\to\infty}\|P_n\|=0$, $\exists$ a natural number k such that

$\|P_n\|<\delta$ for all $n\geq k$

Therefore $\overline{\int_a^b} f(x)\,dx \le U(P_n,f) < \overline{\int_a^b} f(x)\,dx + \varepsilon$ for all $n\geq k$

or, $\overline{\int_a^b} f(x)\,dx - \varepsilon < \overline{\int_a^b} f(x)\,dx \le U(P_n,f) < \overline{\int_a^b} f(x)\,dx + \varepsilon$ $\forall n\geq k$

or, $$\left| U(P_n,f) - \overline{\int_a^b} f(x)\,dx \right| < \varepsilon \quad \forall n \geq k$$

This implies $\lim\limits_{n\to\infty} U(P_n,f) = \overline{\int_a^b} f(x)\,dx$

(ii) Similar proof (exercise)

Worked Examples

1. A function f is defined by $f(x)=x^2$, $x\in[a,b]$, where $a>0$. Find $\overline{\int_a^b} f(x)\,dx$ and $\underline{\int_a^b} f(x)\,dx$. Deduce that f is integrable ~~in~~ on $[a,b]$

Solution: f is bounded on $[a,b]$. Let $P_n = (a, a+h, a+2h, \ldots, a+nh)$ where $h = \frac{b-a}{n}$. Then P_n is a partion of $[a,b]$ dividing $[a,b]$ into n subintervals of equal length.

$\|P_n\| = \frac{b-a}{n}$

Let $M_r = \sup_{x \in [a+(r-1)h,\, a+rh]} f(x)$ $\quad m_r = \inf_{x \in [a+(r-1)h,\, a+rh]} f(x)$, for $r = 1, 2, \ldots, n$

Since f is an increasing function on $[a,b]$

$M_r = (a+rh)^2$, $m_r = (a+(r-1)h)^2$, for $r = 1, 2, \ldots, n$

$$U(P_n, f) = h\left[(a+h)^2 + (a+2h)^2 + \cdots + (a+nh)^2\right]$$

$$= h\left[na^2 + 2ah.\frac{n(n+1)}{2} + h^2.\frac{n(n+1)(2n+1)}{6}\right]$$

$$= nha^2 + a.nh.(nh+h) + \frac{nh(nh+h)(2nh+h)}{6}$$

$$= (b-a)a^2 + a(b-a)^2\left(1+\frac{1}{n}\right) + \frac{1}{6}(b-a)^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$L(P_n, f) = h\left[a^2 + (a+h)^2 + \cdots + (a+(n-1)h)^2\right]$$

$$= h\left[na^2 + 2ah.\frac{n(n-1)}{2} + h^2\frac{n(n-1)(2n-1)}{6}\right]$$

$$= nha^2 + a.nh(nh-h) + \frac{nh(nh-h)(2nh-h)}{6}$$

$$= (b-a)^2 + a(b-a)^2\left(1-\frac{1}{n}\right) + \frac{1}{6}(b-a)^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

Let us consider the sequence of partitions $\{P_n\}$ of $[a,b]$.

$$\lim_{n\to\infty} \|P_n\| = \lim_{n\to\infty} \frac{b-a}{n} = 0$$

Then $\overline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} U(P_n, f)$, $\underline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} L(P_n, f)$

So, $\overline{\int_a^b} f(x)\,dx = (b-a)a^2 + a(b-a)^2 + \frac{1}{6}(b-a)^3 = \frac{b^3 - a^3}{3}$

and $\underline{\int}_a^b f(x)\,dx = (b-a)a^2 + a(b-a)^2 + \frac{1}{6}(b-a)^3 = \frac{b^3-a^3}{3}$

As $\overline{\int}_a^b f(x)\,dx = \underline{\int}_a^b f(x)\,dx$, f is integrable on $[a,b]$

and $\int_a^b f(x)\,dx = \frac{b^3-a^3}{3}$

2. A function f is defined on $[a,b]$ by $f(x) = e^x$.
Find $\overline{\int}_a^b f(x)\,dx$ and $\underline{\int}_a^b f(x)\,dx$. Deduce that f is integrable on $[a,b]$.

Solution: f is bounded on $[a,b]$.
Let $P_n = (a, a+h, a+2h, \ldots, a+nh)$ where $nh = b-a$
Then P_n is a partition of $[a,b]$ into n subintervals of equal length. $\|P_n\| = \frac{b-a}{n}$

Let $M_r = \sup\limits_{x\in[a+(r-1)h,\, a+rh]} f(x)$ and $m_r = \inf\limits_{x\in[a+(r-1)h,\, a+rh]} f(x)$,

for $r = 1, 2, \ldots, n$

Then $M_r = e^{a+rh}$, $m_r = e^{a+(r-1)h}$, $r = 1, 2, \ldots, n$

as e^x is an increasing function.

$$U(P_n, f) = h\left[e^{a+h} + e^{a+2h} + \cdots + e^{a+nh}\right]$$

$$= h.e^{a+h}.\frac{e^{nh}-1}{e^h-1} = h.e^{a+h}.\frac{e^{b-a}-1}{e^h-1}$$

$$= \frac{h.e^h}{e^h-1}\left(e^b - e^a\right)$$

$$L(P_n, f) = h\left[e^a + e^{a+h} + \dots + e^{a+(n-1)h}\right]$$

$$= h \cdot e^a \, \frac{e^{nh}-1}{e^h-1} = \frac{h}{e^h-1} e^a \left(e^{b-a}-1\right)$$

$$= \frac{h}{e^h-1}\left(e^b - e^a\right)$$

Let us consider the sequence of partitions $\{P_n\}$ of $[a,b]$.

$$\lim_{n\to\infty} \|P_n\| = \lim_{n\to\infty} \frac{b-a}{n} = 0$$

Then $\overline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} U(P_n, f)$ and $\underline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} L(P_n, f)$

So, $\overline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} \frac{h e^h}{e^h-1}\left(e^b - e^a\right) = e^b - e^a$

and $\underline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} \frac{h}{e^h-1}\left(e^b - e^a\right) = e^b - e^a$

As $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$, f is integrable on $[a,b]$

and $\int_a^b f(x)\,dx = e^b - e^a$

3. A function f is defined on $[0,1]$ by

$$f(x) = x, \quad x \in [0,1] \cap \mathbb{Q}$$
$$= 0, \quad x \in [0,1] - \mathbb{Q}$$

($\mathbb{Q}$ is the set of all rationals)

Find $\underline{\int_0^1} f(x)\,dx$ and $\overline{\int_0^1} f(x)\,dx$. Deduce that f is not integrable on $[0,1]$

Solution: f is bounded on $[0,1]$. Let us take