the partition P_n of $[0,1]$ defined by

$P_n = \left(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\right)$

Let $M_r = \sup\limits_{x \in \left[\frac{r-1}{n}, \frac{r}{n}\right]} f(x)$, $m_r = \inf\limits_{x \in \left[\frac{r-1}{n}, \frac{r}{n}\right]} f(x)$, $r = 1, 2, \ldots, n$;

Then $M_r = \frac{r}{n}$, $m_r = 0$, for $r = 1, 2, \ldots, n$

$$U(P_n, f) = \frac{1}{n}\left(\frac{1}{n} - 0\right) + \frac{2}{n}\left(\frac{2}{n} - \frac{1}{n}\right) + \cdots + \frac{n}{n}\left(\frac{n}{n} - \frac{n-1}{n}\right)$$
$$= \frac{1}{n}\left[\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n}\right] = \frac{1}{n} \frac{n(n+1)}{2n}$$
$$= \frac{n+1}{2n}$$

and $L(P_n, f) = 0 \cdot \left(\frac{1}{n} - 0\right) + 0 \cdot \left(\frac{2}{n} - \frac{1}{n}\right) + \cdots + 0\left(\frac{n}{n} - \frac{n-1}{n}\right)$
$= 0$

Let us consider the sequence of partitions $\{P_n\}$ of $[0,1]$.

$\|P_n\| = \frac{1}{n}$ and $\lim\limits_{n \to \infty} \|P_n\| = 0$

Then $\overline{\int_0^1} f(x)\,dx = \lim\limits_{n \to \infty} U(P_n, f)$

and $\underline{\int_0^1} f(x)\,dx = \lim\limits_{n \to \infty} L(P_n, f)$

So, $\underline{\int_0^1} f(x)\,dx = 0$ and $\overline{\int_0^1} f(x)\,dx = \frac{1}{2}$

Since $\underline{\int_0^1} f(x)\,dx \neq \overline{\int_0^1} f(x)\,dx$, f is not integrable on $[0,1]$

4. A function f is defined on $[0,1]$ by

$f(x) = x$ if x be rational
$= x^2$ if x be irrational

Find $\underline{\int_0^1} f(x)dx$ and $\overline{\int_0^1} f(x)dx$. Deduce that f is not integrable on $[0,1]$

Solution: f is bounded on $[0,1]$. For all $x \in (0,1)$, $x > x^2$

Let $I = [0,1]$. $f/_{(I \cap \mathbb{Q})}$ is monotonic increasing on $I \cap \mathbb{Q}$

$f/_{(I - \mathbb{Q})}$ is monotonic increasing on $I - \mathbb{Q}$.

Let P_n be the partition of $[0,1]$ defined by $P_n = (x_0, x_1, \cdots, x_n)$,

where $x_0 = 0$, $x_r = \frac{r}{n}$, $r = 1, 2, \cdots, n$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$, for $r = 1, 2, \cdots, n$

Since $f/_{(I \cap \mathbb{Q})}$ is monotonic increasing on $[x_{r-1}, x_r] \cap \mathbb{Q}$,

$$\sup_{x \in [x_{r-1}, x_r] \cap \mathbb{Q}} f(x) = f(x_r) = \frac{r}{n}$$

Since $f/_{(I - \mathbb{Q})}$ is monotonic increasing on $[x_{r-1}, x_r] - \mathbb{Q}$ and x_r is rational, $\sup\limits_{x \in [x_{r-1}, x_r] - \mathbb{Q}} f(x) = \lim\limits_{n \to \infty} f(u_n)$, where $\{u_n\}$ is a sequence of irrational points in $[x_{r-1}, x_r]$ converging to x_r

$$= x_r^2 = \left(\frac{r}{n}\right)^2$$

Since $\frac{r}{n} \geq \left(\frac{r}{n}\right)^2$, $\sup\limits_{x \in [x_{r-1}, x_r]} f(x) = \frac{r}{n}$. Hence $M_r = \frac{r}{n}$ for $r = 1, 2, \cdots, n$

Since $f/_{(I \cap \mathbb{Q})}$ is monotone increasing on $[x_{r-1}, x_r] \cap \mathbb{Q}$,

$$\inf_{x\in[x_{r-1},x_r]} f(x) = f(x_{r-1}) = \frac{r-1}{n}$$

Since $f/_{(I-Q)}$ is monotonic increasing on $[x_{r-1},x_r]-Q$ and x_{r-1} is rational, $\inf_{x\in[x_{r-1},x_r]-Q} f(x) = \lim_{n\to\infty} f(\vartheta_n)$, where $\{\vartheta_n\}$ is a sequence of irrational points in $[x_{r-1},x_r]$ converging to x_{r-1}

$$= x_{r-1}^2 = \left(\frac{r-1}{n}\right)^2$$

Since $\frac{r-1}{n} \geq \left(\frac{r-1}{n}\right)^2$, $\inf_{x\in[x_{r-1},x_r]} f(x) = \left(\frac{r-1}{n}\right)^2$. Hence $m_r = \left(\frac{r-1}{n}\right)^2$, for $r=1,2,\dots,n$

$$U(P_n,f) = \sum_{r=1}^{n} M_r(x_r - x_{r-1})$$

$$= \frac{1}{n}\left[\frac{1}{n}+\frac{2}{n}+\dots+\frac{n}{n}\right] = \frac{n+1}{2n}$$

$$L(P_n,f) = \sum_{r=1}^{n} m_r(x_r - x_{r-1})$$

$$= \frac{1}{n}\left[0+\frac{1^2}{n^2}+\frac{2^2}{n^2}+\dots+\frac{(n-1)^2}{n^2}\right]$$

$$= \frac{(n-1)(2n-1)}{6n^2}$$

Let us consider the sequence of Partion $\{P_n\}$ of $[0,1]$

$\|P_n\| = \frac{1}{n}$ and $\lim_{n\to\infty}\|P_n\| = 0$

Then $\overline{\int_0^1} f(x)dx = \lim_{n\to\infty} U(P_n,f)$ and $\underline{\int_0^1} f(x)dx = \lim_{n\to\infty} L(P_n,f)$

So, $\overline{\int_0^1} f(x)dx = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$ and $\underline{\int_0^1} f(x)dx = \lim_{n\to\infty}\frac{(n-1)(2n-1)}{6n^2} = \frac{1}{3}$

Since $\overline{\int_0^1} f(x)dx \neq \underline{\int_0^1} f(x)dx$, f is not integrable on $[0,1]$.

Theorem 1.4.9 (Another condition of integrability)
Let a function $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$. Then f is integrable on $[a,b]$ if and only if for each $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$U(P,f) - L(P,f) < \varepsilon$ for every partition P of $[a,b]$ satisfying $\|P\| \le \delta$

Proof: Let $f \in \mathcal{R}[a,b]$, i.e., f is Riemann integrable on $[a,b]$. Then $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$

Let us choose $\varepsilon > 0$. Since f is bounded on $[a,b]$, by Darboux theorem, $\exists$ a $\delta_1 > 0$ such that

$U(P,f) < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2}$ for all partition of $[a,b]$ satisfying $\|P\| \le \delta_1$.

Also $\exists$ a $\delta_2 > 0$ such that $L(P,f) > \underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2}$ for all partition P of $[a,b]$ satisfying $\|P\| \le \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$

Then $U(P,f) < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2}$ and $L(P,f) > \underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2}$ for all partition P of $[a,b]$ satisfying $\|P\| < \delta$

As $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$ (say)

So, $U(P,f) - L(P,f) < \varepsilon$ for all partition P of $[a,b]$ satisfying $\|P\| \le \delta$.

To prove the converse, we first observe that for