any partition P of $[a,b]$,

$$L(P,f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(P,f)$$

That is, $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \le U(P,f) - L(P,f)$

for any partition P of $[a,b]$

Let us choose $\varepsilon>0$. By the condition, $\exists\ \delta>0$ such that for all partition P of $[a,b]$ satisfying $\|P\| \le \delta$, $U(P,f) - L(P,f) < \varepsilon$ holds.

Therefore, $\exists$ a partition, say, P_ε of $[a,b]$ such that $\|P_\varepsilon\| \le \delta$ and $U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$

Also, we have $\overline{\int_a^b} f(x)\,dx \ge \underline{\int_a^b} f(x)\,dx$ by Theorem 1.4.3.

Therefore, $0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$.

This holds for each positive number ε. It follows that $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$ and hence f is integrable on $[a,b]$.

This completes the proof.

1.5. Concept of Negligible set (or Zero set) (or a set of measure zero)

1.5.1. Definition: A set $S \subset \mathbb{R}$ is said to be a Negligible set or a Zero set or a set of measure zero if for each $\varepsilon>0$ $\exists$ a countable collection of open intervals $\{I_n\}$ such that $S \subset \bigcup_{n\ge1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| < \varepsilon$ where

$|I_n| = b_n - a_n$ = length of the open interval $I_n \left(= (a_n, b_n)\right)$

Examples: 1. Let B be a negligible set and $A \subset B$.

we show that A is also ~~both~~ negligible set.

As B is a negligible set, $\exists$ a countable collection of intervals $\{I_n\}$ such that $B \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| < \varepsilon$ for ~~some~~ each $\varepsilon > 0$.

As $A \subset B$ So, for the countable collection $\{I_n\}$,

$$A \subset B \subset \bigcup_{n=1}^{\infty} I_n \quad \text{and} \quad \sum_{n=1}^{\infty} |I_n| < \varepsilon, \text{ for each } \varepsilon > 0.$$

So, A is negligible

2. A finite set $S \subset \mathbb{R}$ is a negligible set

Let $S = \{x_1, x_2, \dots, x_m\} \subset \mathbb{R}$

Let $\varepsilon > 0$. Consider the open interval

$$I_r = \left(x_r - \frac{\varepsilon}{2(m+1)}, x_r + \frac{\varepsilon}{2(m+1)}\right). \text{ So, } x_r \in I_r, \ r = 1, 2, \dots, m$$

Then $S \subset \bigcup_{r=1}^{m} I_r$ and $|I_1| + |I_2| + \dots |I_m|$

$$= m \cdot \frac{\varepsilon}{m+1} < \varepsilon$$

So, S is covered by a finite collection of open intervals such that the sum of the length of the intervals is less than ε, proving that S is a negligible set.

3. Let A_k, $k = 1, 2, \dots$, be a countable collection of negligible sets. Let $A = \bigcup_{k=1}^{\infty} A_k$. Then we show that A is a negligible set. As A_k, $k = 1, 2, \dots$ are negligible set $\exists$ collection of open intervals $\{I_{nk}\}_n$ such that $A_k \subset \bigcup_{n=1}^{\infty} I_{nk}$ and $\sum_{n=1}^{\infty} |I_{nk}| < \frac{\varepsilon}{2^k}$ for each $\varepsilon > 0$, $k = 1, 2, \dots$

Now consider the collection $\{I_{nk} : n=1,2,\cdots, k=1,2,\cdots\}$ open intervals.

Then $\bigcup_{k=1}^{\infty} A_k \subset \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} I_{nk}$ $\left[\text{As } A_k \subset \bigcup_{n=1}^{\infty} I_{nk},\ k=1,2,\cdots\right]$

Also $\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |I_{nk}| < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \frac{\varepsilon}{2}\left[1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right]$

$= \frac{\varepsilon}{2}\,\frac{1}{1-\frac{1}{2}} = \varepsilon$

So, $\bigcup_{k=1}^{\infty} A_k$ is a negligible set.

4. A countable subset of $\mathbb{R}$ is a negligible set.

Let S be a countable set. Let $S = \{x_1, x_2, \cdots\cdots\}$

Let $\varepsilon>0$. For each r, let $I_r = \left(x_r - \frac{\varepsilon}{2^{r+2}},\ x_r + \frac{\varepsilon}{2^{r+2}}\right)$

Then $x_r \in I_r$, $r=1,2,\cdots$

Then $S \subset \bigcup_{r=1}^{\infty} I_r$

Now $\sum_{r=1}^{\infty} |I_r| = \frac{\varepsilon}{2^2}+\frac{\varepsilon}{2^3}+\cdots$

$= \frac{\varepsilon}{2^2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)$

$= \frac{\varepsilon}{2^2}\cdot 2 = \frac{\varepsilon}{2} < \varepsilon$

So, S is a negligible set.

Corollary: Since $\mathbb{Q}$, the set of all rational numbers is a countable set, it is a negligible set.

5. Let S be a bounded infinite subset of $\mathbb{R}$

having a finite number of limit points. Then S is a negligible set.

Let the limit points of S be $x_1, x_2, \dots, x_m$

Let $I_r = \left(x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2}\right)$, $r = 1, 2, \dots m$

and choose δ_r, $r = 1, 2, \dots, m$ such that

$$\sum_{r=1}^{m} \delta_r < \frac{\varepsilon}{2}$$

outside the open intervals I_r, $r = 1, 2, \dots, m$ there lie at most a finite number of points of S and those can be covered by a finite number of open intervals, the sum of whose lengths is less than $\frac{\varepsilon}{2}$

Considering the two finite families of open intervals, S is covered by a ~~collection~~ countable collection of open intervals, the sum of whose lengths is less than ε

Therefore S is a negligible set.

In this respect, we state a theorem of Lebesgue without proof:

1.5.2. Theorem : A bounded function on closed and bounded interval is Riemann integrable if and only if the set of points of discontinuity is negligible.