rational

point, $f(x) \geq \frac{1}{k} > \frac{\varepsilon}{2}$ and at all other points, $f(x) \leq \frac{\varepsilon}{2}$. Let the finite number of rational points for which $f(x) > \frac{\varepsilon}{2}$ be $x_1, x_2, \ldots, x_m$ where

$x_1 < x_2 < \ldots < x_m$

Let us enclose the points by subintervals $[x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}]$, $[x_2 - \frac{\delta_2}{2}, x_2 + \frac{\delta_2}{2}], \ldots, [x_m - \frac{\delta_m}{2}, x_m + \frac{\delta_m}{2}]$ such that

$\delta_1 + \delta_2 + \cdots + \delta_m < \frac{\varepsilon}{2}$

Since each of these subintervals contain rational as well as irrational points, the oscillation of f in each ~~of~~ of these intervals is less than 1

~~Let $P = (0, x_1 - \frac{\delta_1}{2}, x_2 + \frac{\delta_2}{2}, x_3 - \frac{\delta_3}{2}$~~

Let $P = \left(0, x_1 - \frac{\delta_1}{2}, x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}, x_2 \ldots, x_m + \frac{\delta_m}{2}, 1\right)$. Then P is a partition of $[0,1]$ dividing $[0,1]$ into $2m+1$ subintervals, m of which enclose the points $x_1, x_2, \ldots, x_m$. In each of the remaining $(m+1)$ subintervals, the oscillation of f is less than $\frac{\varepsilon}{2}$ and sum of these subintervals is less than 1.

So, $U(P,f) - L(P,f) < 1 \cdot \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \cdot 1 = \varepsilon$

Therefore, $\exists$ a partition P of $[0,1]$ such that $U(P,f) - L(P,f) < \varepsilon$. This being a sufficient condition for integrability, f is integrable on $[0,1]$

Let $P = (x_0, x_1, \ldots, x_n)$, where $0 = x_0 < x_1 < \cdots < x_n = 1$ be an arbitrary partition of $[0,1]$. Let

$m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$, $r = 1, 2, \ldots, n$

Since every subinterval $[x_{r-1}, x_r]$ contains irrational points, $m_r = 0$ for $r = 1, 2, \ldots, n$

Therefore $L(P, f) = 0$

Consequently, $\underline{\int_0^1} f(x)\,dx = \sup\{L(P,f) : P \in \mathcal{P}[a,b]\}$
$= 0$

Since $f \in \mathcal{R}[0,1]$, $\underline{\int_0^1} f(x)\,dx = \int_0^1 f(x)\,dx$

and therefore $\int_0^1 f(x)\,dx = 0$

Note: The function f is continuous at 0 and at every irrational point in $[0,1]$ and discontinuous at every non-zero rational points of $[0,1]$. This example shows that a function bounded and continuous on a closed and bounded interval $[a,b]$ except for an infinite set of points $S \subset [a,b]$ having infinite number of limit points, is also Riemann integrable.

Remarks: We have seen that a bounded function f

(i) continuous on a closed interval $[a,b]$ is integrable on $[a,b]$

(ii) continuous on $[a,b]$ except at a finite number of points is integrable on $[a,b]$

(iii) continuous on $[a,b]$ with infinite number of points of discontinuities is integrable on $[a,b]$ provided the set of points of discontinuity has only a finite number of limit points.

we have proved that (iii) is not the most general cases by ~~an~~ example 5.

In fact, we have earlier stated the most general Theorem of Lebesgue without proof which is Theorem 1.5.2.

1.7 Properties of Riemann integrable functions.

Theorem 1.7.1 Let $f:[a,b]\to\mathbb{R}$, $g:[a,b]\to\mathbb{R}$ be both integrable on $[a,b]$. Then $f+g$ is integrable on $[a,b]$ and $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

Proof: Since $f,g\in\mathcal{R}[a,b]$, f and g are both bounded on $[a,b]$. So, $\exists$ positive real numbers k_1, k_2 such that

$|f(x)| < k_1$ and $|g(x)| < k_2$ $\forall x\in[a,b]$

So, $|f(x)+g(x)| \le |f(x)| + |g(x)| < k_1+k_2$ $\forall x\in[a,b]$

This shows that $f+g$ is bounded on $[a,b]$

Let us choose $\varepsilon>0$

Since $f\in\mathcal{R}[a,b]$, $\exists$ a partition P_1 of $[a,b]$ such that

$$U(P_1,f) - L(P_1,f) < \varepsilon/2$$

Since $g\in\mathcal{R}[a,b]$, $\exists$ a partition P_2 of $[a,b]$ such that

$$U(P_2,g) - L(P_2,g) < \frac{\varepsilon}{2}$$

Let $P_0 = P_1\cup P_2$. Then P_0 is a refinement of P_1 as well as P_2 and $L(P_1,f)\le L(P_0,f)\le U(P_0,f)\le U(P_1,f)$

$$L(P_2, g) \le L(P_0, g) \le U(P_0, g) \le U(P_1, g)$$

So, $U(P_0, f) - L(P_0, f) \le U(P_1, f) - L(P, f) < \varepsilon/2$

and $U(P_0, g) - L(P_0, g) \le U(P_2, g) - L(P_2, g) < \varepsilon/2$

Let $P_0 = (x_0, x_1, \ldots, x_n)$, where $a = x_0 < x_1 < x_2 \cdots < x_n = b$

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} (f+g)(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} (f+g)(x)$

$M_r' = \sup\limits_{x \in [x_{r-1}, x_r]} f(x)$, $m_r' = \inf\limits_{x \in [x_{r-1}, x_r]} f(x)$

$M_r'' = \sup\limits_{x \in [x_{r-1}, x_r]} g(x)$, $m_r'' = \inf\limits_{x \in [x_{r-1}, x_r]} g(x)$

Then $M_r \le M_r' + M_r''$, $m_r \ge m_r' + m_r''$, for $r = 1, 2, \ldots, n$

$$\begin{aligned} U(P_0, f+g) &= M_1(x_1 - x_0) + \cdots + M_n(x_n - x_{n-1}) \\ &\le [M_1'(x_1 - x_0) + \cdots + M_n'(x_n - x_{n-1})] + [M_1''(x_1 - x_0) + \cdots + M_n''(x_n - x_{n-1})] \\ &= U(P_0, f) + U(P_0, g) \end{aligned}$$

Similarly $L(P_0, f+g) \ge L(P_0, f) + L(P_0, g)$

$$\begin{aligned} \text{So, } U(P_0, f+g) - L(P_0, f+g) &\le U(P_0, f) + U(P_0, g) - (L(P_0, f) + L(P_0, g)) \\ &= (U(P_0, f) - L(P_0, f)) + (U(P_0, g) - L(P_0, g)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore, for a chosen $\varepsilon > 0$, $\exists$ a ~~partition~~ partition P_0 of $[a, b]$ such that

$$U(P_0, f+g) - L(P_0, f+g) < \varepsilon$$