Notes on DSEB for SEMESTER VI

(Point Set Topology)

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Books followed : 1. Topology, A first course - J. R. Munkres

2. Topology - J. Dugundji

3. Introduction to General Topology - K. D. Joshi

4. Elements of General Topology - S. T. Hu.

Syllabus

Unit-1 Topological spaces, basis and subbasis for a topology, neighbourhoods of a point, interior points, limit points, derived set, boundary of a set, closed sets, closure and interior of a set, dense subsets, subspace topology, finite Product topology, Continuous functions, open maps, closed maps, homeomorphisms, topological invariants, metric topology, isometry and metric invariants

Unit-2 First countability, T_1 and T_2 separation axioms of topological spaces. Convergence and cluster point of a sequence in topological spaces and some related concepts on first countable as well as on T_2 space. Heine's Continuity criterion.

Unit-3 Connected spaces, Connected sets in $\mathbb{R}$, components. Compact spaces, compactness and T_2, compact sets in $\mathbb{R}$. Heine Borel Theorem for $\mathbb{R}^n$, real valued continuous function on connected and compact spaces, the concept of compactness in metric space, sequential compactness of a metric space X and the Bolzano Weierstrass property of X are equivalent.

1. Topological Spaces

1.1 Definition : A family $\mathfrak{I}$ of subsets of a non-empty set X is said to be a topology on X if the following conditions are satisfied:

(i) $\phi, X \in \mathfrak{I}$

(ii) If I be an arbitrary index set such that $G_\alpha \in \mathfrak{I}$ for all $\alpha \in I$, then $\bigcup_{\alpha \in I} G_\alpha \in \mathfrak{I}$ (i.e., $\mathfrak{I}$ is closed under arbitrary unions)

(iii) If $G_1, G_2 \in \mathfrak{I}$ then $G_1 \cap G_2 \in \mathfrak{I}$ (Hence if $G_i \in \mathfrak{I}$, $i = 1, 2, \dots, n$ then $\bigcap_{i=1}^{n} G_i \in \mathfrak{I}$) (i.e., $\mathfrak{I}$ is closed under finite intersections)

The pair $(X, \mathfrak{I})$ is called a topological space.
Any set $G \in \mathfrak{I}$ is said to be a open set of X.
As many topologies can be defined on the same set, so, if $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are two topologies on X, then open sets in ~~topolg~~ topological space $(X, \mathfrak{I}_1)$ and topological space $(X, \mathfrak{I}_2)$ are different.

Note: We denote a topology by $\mathfrak{I}$ or τ or T etc.

Examples of Topological Spaces: 1. Let X be a non-empty set. Then $\tau = \{\phi, X\}$ is a topology on X, called the indiscrete topology or trivial topology.

2. Let X be any non-empty and τ be the collection of all subsets of X. Then it is clearly a

topology for X. It is called the discrete topology.

3. On a two point set $X = \{a, b\}$, $\tau = \{\phi, \{a\}, X\}$ forms a topology on X. It is neither discrete nor indiscrete. ~~It is sometimes~~ ~~called~~ (X, τ) is sometimes called Sierpinski space and τ is called Sierpinski topology on X.

4. On the set $X = \{a, b, c\}$, several topologies are possible.

$\mathcal{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ is one such topology.

5. Let X be a set; let τ be the collection of all subsets U of X such that $X - U$ either is finite or is all of X. Then τ is a topology on X, called the **cofinite** topology or finite complement topology. Both X and ϕ are in τ, since $X - X = \phi$ is finite and $X - \phi$ is all of X. If $\{U_\alpha : \alpha \in I\}$ is an indexed collection of elements of τ, to show that $\bigcup_{\alpha \in I} U_\alpha$ is in τ, we compute

$$X - \bigcup_{\alpha \in I} U_\alpha = \bigcap_{\alpha \in I} (X - U_\alpha).$$ The latter set is finite because each set $X - U_\alpha$ is finite.

If $U_1, U_2, \ldots, U_n$ are elements of τ, to show that $\bigcap_{i=1}^{n} U_i$ is in τ, we compute $X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$. The latter set is a finite union of finite sets and is therefore finite.

Example 6. Let X be a set. Let $\mathcal{T}$ be the collection of all subsets U of X such that $X - U$ is either countable or is all of X. Then $\mathcal{T}$ is a topology on X (Check it). $\mathcal{T}$ is called cocountable topology on X.

1.2 Given two topologies τ_1 and τ_2 on the same set X, we say τ_1 is weaker (smaller, coarser) than τ_2, or τ_2 is stronger (larger, finer) than τ_1 if $\tau_1 \subset \tau_2$.

1.3 (Definition). Let (X, τ) be a topological space, we say a set $F \subset X$ is closed if

$$X - F \in \tau.$$

Instead of (X, τ), we write the topological space as X when no confusion arises about the topology. So, sometimes we write, A set F in a topological space X is closed if $X - A$ is open.

Theorem 1.4 If $\mathcal{F}$ is the collection of all closed sets in a topological space X, then

(i) $\bigcap_{\alpha \in I} F_\alpha \in \mathcal{F}$ for any arbitrary indexed collection $\{F_\alpha : \alpha \in I\}$ of closed sets in $\mathcal{F}$.

(ii) If $F_1, F_2, \ldots, F_n \in \mathcal{F}$ then $\bigcup_{i=1}^{n} F_i \in \mathcal{F}$

(iii) X and ϕ both belong to $\mathcal{F}$

Conversely, given a set X and any family $\mathcal{F}$ of subsets of X satisfying (i), (ii) and (iii), the collection of