basis because $\prod_{\alpha\in J} X_\alpha$ is itself a basis element; and it satisfies the second condition because the intersection of any two basis elements is another basis element:

$$\left(\prod_{\alpha\in J} U_\alpha\right) \cap \left(\prod_{\alpha\in J} V_\alpha\right) = \prod_{\alpha\in J} (U_\alpha \cap V_\alpha)$$

This topology is not the most useful one for the product space $\prod X_\alpha$, as we shall see.

A second way to generalize the previous definition is to generalize the subbasis formulation of the definition. Let

$$\pi_\beta : \prod_{\alpha\in J} X_\alpha \to X_\beta$$ be the

function assigning to each element of the product space its βth coordinate, $\pi_\beta((x_\alpha)_{\alpha\in J}) = x_\beta$; it is called the projection mapping associated with the index β.

<u>Definition</u> Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ open in } X_\beta\}$$

and let $\mathcal{S}$ denote the union of these collections,

$$\mathcal{S} = \bigcup_{\beta\in J} \mathcal{S}_\beta$$

The topology generated by the subbasis $\mathcal{S}$ is called the product topology. In this ... a product space:

How does the product topology differ from the box topology;

It is easier to answer this question if we look at the basis $\mathcal{B}$ that $\mathcal{S}$ generates. The collection $\mathcal{B}$ consists of all finite intersections of elements of $\mathcal{S}$.

If we intersect elements belonging to the same one of the sets $\mathcal{S}_\beta$ we do not get anything new, because

$$\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \pi_\beta^{-1}(U_\beta \cap V_\beta);$$

the intersection of two elements of $\mathcal{S}_\beta$, or of finitely many such elements, it again is an element of $\mathcal{S}_\beta$. We get something new only when we intersect elements from different sets $\mathcal{S}_\beta$. The typical element of the basis $\mathcal{B}$ can thus be described as follows: Let $\beta_1, \beta_2, \ldots, \beta_n$ be a finite set of distinct indices from the index set J, and let U_{β_i} be an open set in X_{β_i}, for $i=1,2,\ldots,n$.

Then $B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$

is an element of $\mathcal{B}$.

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There is another way to describe this basis element which is particularly useful. Note that a point $x = (x_\alpha)$ is in B if and only if its β_1th coordinate is in U_{β_1}, its β_2th coordinate is in U_{β_2}, and so on. There is no restriction on the αth coordinate of x if α is not one of the indices $\beta_1, \beta_2, \ldots, \beta_n$. As a result, we can write B as the product

$$B = \prod_{\alpha \in J} U_\alpha,$$

where U_α denotes the entire space X_α if $\alpha \neq \beta_1, \ldots, \beta_n$.

All this is summarized in the following theorem:

Theorem 7.4 (Comparison of the box and product topologies): The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α.

Two things are immediately clear. First, for

First, for finite products $\prod_{\alpha=1}^{n} X_\alpha$ the two topologies are precisely the same. Second, the box topology is in general finer than the product topology.

What is not clear is why we prefer the product topology to the box topology. The answer will appear as we continue our study of topology. We will find that a number of important theorems about finite products will also hold for arbitrary products if we use the product topology, but not if we use the box topology. As a result, the product topology is extremely important in Mathematics. The box topology is not so important; we shall use it primarily for constructing counterexamples. Therefore, wheaever whenever we consider the product $\prod_\alpha X_\alpha$, we shall assume that it is given the product topology unless we specifically state otherwise.

Some of the theorems we proved for the product $X \times Y$ hold for the product $\prod X_\alpha$ no matter which topology we use. We list them here: