Theorem 7.5 Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form $\prod_{\alpha\in J} B_\alpha$ where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha\in J} X_\alpha$.

The collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology on $\prod_{\alpha\in J} X_\alpha$.

Theorem 7.6 Let A_α be a subspace of X_α, for each $\alpha\in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both products are given the product topology.

~~Theorem 7.7~~ The proofs of these theorems follow the pattern of the proofs already given for $X\times Y$, so the details are left to you as exercise.

8. Continuity and related concepts

8.1 Definition: Let $f: X \to Y$ be a function; $x_0 \in X$ and $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X, Y respectively.

Then f is said to be continuous at x_0 if for every $V \in \mathcal{T}_2$ such that $f(x_0) \in V$, $\exists\ U \in \mathcal{T}_1$ such that $x_0 \in U$ and $f(U) \subset V$. We say f is continuous on X iff f is continuous at each $x_0 \in X$

Theorem 8.2 If X and Y are topological spaces and $f : X \to Y$, then the following are equivalent

(a) f is continuous at x_0

(b) for each neighbourhood M of $f(x_0)$ in Y, $\exists$ a neighbourhood N of x_0 such that $f(N) \subset M$

Proof: (a) $\Rightarrow$ (b). Let M be a nbd of $f(x_0)$. Then $\exists$ V open in Y such that $f(x_0) \in V \subset M$. As f is continuous at x_0, so $\exists$ open set U in X such that $x_0 \in U$ and $f(U) \subset V \subset M$

Take $N = U$. So $f(N) \subset M$.

So, (a) $\Rightarrow$ (b)

now to prove (b) $\Rightarrow$ a, let $f(x_0) \in V$, V an open subset of Y. So, V is a nbd of $f(x_0)$. So, $\exists$ a nbd N ~~such that~~ of x_0 such that $f(N) \subset V$ ...(i)

As $x_0 \in N$ and N is a nbd of x_0, $\exists$ an open set U such that $x_0 \in U \subset N$

Now $f(U) \subset f(N) \subset V$ (by (i))

So, (b) $\Rightarrow$ (a).

Theorem 8.3 If X and Y are topological spaces and $f : X \to Y$, then the following are all equivalent:

(a) f is continuous

(b) for each open set H in Y, $f^{-1}(H)$ is open in X

(c) for each closed set K in Y, $f^{-1}(K)$ is closed in X

(d) for each $E \subset X$, $f(Cl_X E) \subset Cl_Y f(E)$

Proof: a) $\Rightarrow$ b) If H is open in Y, then for each $x \in f^{-1}(H)$, $f(x) \in H$. As f is continuous in X, so it is continuous on x.

So, $\exists$ open set U_x, such that $x \in U_x$ and $f(U_x) \subset H$ or, $U_x \subset f^{-1}(H)$

and $f^{-1}(H) = \bigcup_{x \in f^{-1}(H)} U_x$. So, $f^{-1}(H)$ is open in X.

(b) $\Rightarrow$ (c): If K is closed in Y, then $f^{-1}(Y-K)$ is open in X, by part (b). Hence $f^{-1}(K) = X - f^{-1}(Y-K)$, $f^{-1}(K)$ is closed in X.

(c) $\Rightarrow$ (d): Let K be any closed set in Y containing $f(E)$. By part (c), $f^{-1}(K)$ is a closed set in X containing E. Hence $cl_X E \subset f^{-1}(K)$, and it follows that $f(cl_X E) \subset K$. Since this is true for any closed set K containing E, we have

$$f(cl_X E) \subset cl_Y f(E)$$

(d) $\Rightarrow$ (a): Let $x \in X$ and let V be any open set containing $f(x)$. Set $E = X - f^{-1}(V)$ and let $U = X - cl_X E$. Now from (d)

$$f(cl_X E) \subset cl_Y f(E) \quad \text{--- (1)}$$

Now since (1) holds $x \in U$ and $f(U) \subset V$ (prove it). Hence f is continuous at x. As $x \in X$ is arbitrary, so f is continuous on X.

Theorem 8.4: If X, Y and Z are topological spaces and $f: X \to Y$, and $g: Y \to Z$ are continuous, then $g \circ f: X \to Z$ is continuous