Proof: If H is open in Z, then $g^{-1}(H)$ is open in Y, by continuity of g. Hence, by continuity of f, $f^{-1}[g^{-1}(H)) = (g \circ f)^{-1}(H)$ is open in X. Thus $g \circ f$ is continuous.

8.5 Definition: If $f : X \to Y$ and $A \subset X$, we shall use $f/_A$ (the restriction of f to A) to denote the map of A into Y defined by $f/_A(a) = f(a)$ for each $a \in A$.

8.6 If $A \subset X$ and $f : X \to Y$ is continuous, then $f/_A : A \to Y$ is continuous.

Proof: If H is open in Y, then $(f/_A)^{-1}(H) = f^{-1}(H) \cap A$, and the latter is open in the relative topology on A as $f^{-1}(H)$ is open in X (since f is continuous).

~~8.8~~ Theorem 8.7 If $X = A \cup B$ where A and B are both open (or both closed) in X, and if $f : X \to Y$ is a function such that both $f/_A$ and $f/_B$ are continuous, then f is continuous.

Proof: Suppose A and B are open. If H is open in Y, then $f^{-1}(H)$ is open in X, since $f^{-1}(H) = (f/_A)^{-1}(H) \cap (f/_B)^{-1}(H)$

and each of the latter is open in ~~B~~ an open subspace of X and so open in X. The proof is similar if A and B are closed sets.

Theorem 8.8 Suppose $Y \subset Z$ and $f: X \to Y$. Then f is continuous as a map from X to Y iff it is ~~a~~ continuous as a map from X to Z.

Proof: Exercise.

8.9. Definition: Let X, Y be two topological spaces. A function $f: X \to Y$ is said to be open (respectively closed) if whenever A is open ~~in X~~, (resp. closed) ~~in~~ ~~B~~ in X, $f(A)$ is open (resp. closed) in Y

8.10 Definition: Let X, Y be two topological space. A function $f: X \to Y$ is said to be a homeomorphism from X to Y if f is a bijection and both f and f^{-1} are continuous. When such a homeomorphism exists, X is said to be homeomorphic to Y.

Theorem 8.11: Let X, Y be topological spaces and $f: X \to Y$ a function. Then the following statements are equivalent:

(1) f is a homeomorphism

(2) f is a continuous bijection and f is open

(3) f is a bijection and f^{-1} is an open map.

(4) if $G \subset X$, then $f(G)$ is open in Y iff G is open in X

(5) if $F \subset X$, then $f(F)$ is closed in Y iff F is closed in X.

(6) if $E \subset X$, then $f(cl_X E) = cl_Y f(E)$

(7) $\exists$ a function $g: Y \to X$ such that f, g are continuous, $g \circ f = i_X$ and $f \circ g = i_Y$ where i_X and i_Y are identity function on X and Y respectively.

Proof: Use Theorem 8.3 to prove the results.

8.12 ~~Theorem~~ Definition: Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two topological spaces. ~~A function is said~~

A function $f: X \to Y$ is said to be an embedding (or imbedding) of X into Y if f is a homeomorphism when regarded as a function from $(X, \mathcal{T}_1)$ onto $(f(X), \mathcal{T}_{f(X)})$

Theorem 8.13 A function $f: X \to Y$ is an embedding iff it is continuous and one-to-one and for every open

set V in X, $\exists$ an open subset W of Y such that
$f(V) = W \cap Y$

Proof: The result follows directly from the definition of a homeomorphism and of relative topology.

Note: Homeomorphic topological spaces are, for the purpose of a topologist, the same. That is, there is nothing about homeomorphic spaces X and Y having to do only with their respective topologies which we can use to distinguish them. Thus, for example, a "topological characterization" of the real line $\mathbb{R}$ would consist of a list of properties possessed by the real line which, if possessed by any other space X, ensure that X is homeomorphic with $\mathbb{R}$.

If we denote "X is homeomorphic with Y" by $X \sim Y$, then the relationship $\sim$ has the following properties:

a) $X \sim X$

b) if $X \sim Y$ then $Y \sim X$

(c). if $X \sim Y$, $Y \sim Z$ then $X \sim Z$

Thus, the relation "is homeomorphic to" is an equivalence relation on any set of topological spaces.

To prove two spaces are homeomorphic, one constructs a