homeomorphism. To establish that two spaces are not homeomorphic, one must find a topological property possessed by one and not the other. The definition of "topological property" makes it clear why this works. A topological property is a property of topological spaces, which if possessed by X, is possessed by all spaces homeomorphic to X.

8.12.1 Examples: a) The open interval (a, b) in $\mathbb{R}$ is homeomorphic to $(0,1)$, one homeomorphism being $f(x) = \frac{x-a}{b-a}$. Moreover, all intervals of the form (a, ∞) are obviously homeomorphic by translation, and $(1, \infty)$ is homeomorphic to $(0,1)$ under the map $f(x) = \frac{1}{x}$. Also the interval $(-\infty, -a)$ is homeomorphic to (a, ∞) under the map $f(x) = -x$. Finally $(-\infty, \infty)$ is homeomorphic to $(-\pi/2, \pi/2)$ under the map $f(x) = \tan^{-1} x$. The relations above can be summarized, using transitivity of the homeomorphism relation, as follows: all open intervals in $\mathbb{R}$, including the unbounded intervals, are homeomorphic.

b) All bounded closed intervals which have more than one point are homeomorphic. In fact $[a, b]$ is homeomorphic to $[0,1]$ under the same map $f(x) = \frac{x-b}{b-a}$ used above.

8.13. Metric topology

Let (X, d) be a metric space. A subset A of X is said to be open in X if for each $x \in A$, $\exists$ an open ball $B_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$ for some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A$. Let $\tau = \{A : A \text{ is open in } X\}$. Then τ is a topology on X, called the metric topology induced by the metric d on X.

8.13.1 Isometry

Let (X, d_1) and (Y, d_2) be two metric spaces. A mapping $f : X \to Y$ is said to be an isometry if for all $x, y \in X$, $d_2(f(x), f(y)) = d_1(x, y)$

Metric spaces (X, d_1) and (Y, d_2) are said to be isometric if $\exists$ a bijective mapping $f : X \to Y$ such that f is an isometry.

We can prove that if f is an isometry then both f and f^{-1} are continuous functions.

So, isometric space "metrically identical"; that is, there is nothing about their respective metric which will serve to distinguish them.

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9. First countability

9.1. Definition: Let X be a topological space and $x \in X$. Then a local base at x is a collection $\mathcal{B}$ of neighbourhoods of x such that given any neighbourhood N of x, $\exists$ $B \in \mathcal{B}$ satisfying $B \subset N$.

For example, the collection of all open neighbourhoods of a point is a local base at that point. If X is a metric space, then the collection of open balls (closed balls will do as well) centred at a point constitute a local base at that point. Note that it suffices to take open balls of rational radii, thereby giving a local base which is countable.

9.2 Definition: A topological space is said to be first countable at a point if $\exists$ a countable local base at that point

9.3 Definition: A topological space is said to be first countable or to satisfy the first axiom of countability if it is first countable at each point.

As we noted just before giving the definitions, all metric spaces are first countable.

9.3. Definition: A topological space X is said to satisfy

T_1-axiom or is said to be T_1-space if for every two distinct points x and y in X, $\exists$ an open set containing x but not y (and hence also another open set containing y but not x).

All the metric spaces are T_1-spaces.

9.4 Theorem: For a topological space $(X, \mathcal{T})$ the following are equivalent:

(1) The space X is a T_1-space

(2) For any $x \in X$, the singleton set $\{x\}$ is closed.

(3) Every finite subset of X is closed

(4) The topology $\mathcal{T}$ is stronger than the cofinite topology

Proof: ~~As~~ Only the equivalence of (1) with (2) needs to be established. The rest follow from the properties of closed sets and the definition of cofinite topology. Assume (1) holds and let $x \in X$, we claim that $X - \{x\}$ is a neighbourhood of each of its points. For, let $y \in X - \{x\}$. Then x, y are distinct points of X and so by the T_1-axiom, $\exists$ an open set, say, V such that $y \in V$ and $x \notin V$ But this means $V \subset X - \{x\}$ and hence $X - \{x\}$