is a neighbourhood of y. So, $X - \{x\}$ is open and therefore $\{x\}$ is closed in X. Conversely assume (2). Let x, y be distinct points of X. Then $X - \{y\}$ is an open set which contains x but not y.

Thus T_1-axiom holds in X.

In view of the theorem above, the cofinite topology on a set X is the weakest topology which makes X into a T_1-space. T_1-spaces share some of the pleasent properties of metric spaces. As an illustration, we prove the following simple proposition whose truth in the case of metric space must already be known to the reader.

<u>9.5 Theorem</u> : Suppose y is a limit point of a subset A of a T_1 space. Then every neighbourhood of y contains infinitely many points of A.

Proof: Let N be a neighbourhood of y and let $F = A \cap N - \{y\}$. We claim that the set F is infinite. For, if not, $X - F$ will be an open set containing y and so $N \cap X - F$ will be a neighbourhood of y. Evidently this neighbourhood contains no points of A, except possibly y, contradicting that y is a limit point of A. So F is infinite, showing that every neighbourhood of y contains infinitely many points of A.

9.6. Definition: Let $\{x_n\}$ be a sequence in a topological space $(X, \mathcal{T})$. $\{x_n\}$ is said to converge to a point y of X if for every open set U containing y, $\exists$ a positive integer N such that for every integer $n \geq N$, $x_n \in U$. We write $x_n \to y$ as $n \to \infty$.

Despite their nice properties, T_1 spaces are not totally free of anomalous behaviour, the most serious of which is connection with convergence of sequences. In a T_1 space, a sequence may converge to more than one point. In fact, it may converge to every point of the space as in the case of the cofinite topology on an infinite set. This is certainly inconsistent with our expectation that limit of a sequence be unique, a natural requirement since so many things are defined as limits of something.

Uniqueness of limits holds in spaces which have the Hausdorff property, which we now formally state as our next separation axiom.

9.7 Definition A topological space $(X, \mathcal{T})$ is said to satisfy T_2 axiom (or the Hausdorff property) or is said to be a T_2 space (or Hausdorff space) if for ~~every~~ each pair of distinct points $x, y \in X$, $\exists$ disjoint open sets U, V in X such that $x \in U$ and $y \in V$.

Evidently, every T_2 space is a T_1 space since the condition in the definition implies that U contains x but not y and that V contains y but not x. The converse is false. An infinite set with cofinite topology is T_1 but not T_2. In fact, no two open sets in it are disjoint unless one of them is empty.

All metric spaces are T_2 (prove it)

9.8. Theorem In a T_2 or Hausdorff space, limits of sequences are unique.

Proof: Let $\{x_n\}$ be a sequence in a topological space X. and suppose $x_n \to x$ and $x_n \to y$ as $n \to \infty$. We have to show that $x = y$. If not, $\exists$ open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Then $\exists$ $N_1, N_2 \in \mathbb{N}$ such that $x_n \in U$ ~~and~~ $\forall$ $n \geq N_1$ and $x_n \in V$ for all $n \geq N_2$. Let m be a positive integer greater than both N_1 and N_2. Then $x_n \in U \cap V$ $\forall$ $n \geq m$, contradicting that $U \cap V = \phi$. So, $x = y$. Thus limits of sequences in a T_2 space, ~~are unique~~ when they exist, are unique.

This theorem does not say anything about the existence of limit of ~~a~~ a sequence in a T_2 space. It only says that if limit of a sequence exists in a T_2 space, then it is unique.

<u>9.9 Theorem</u>: Let X, Y be topological spaces, $x \in X$ and $f: X \to Y$ a function. Suppose X is first countable at x. Then, f is continuous at x if and only if for every sequence $\{x_n\}$ ~~in~~ which converges to x in X, the sequence $\{f(x_n)\}$ converges to $f(x)$ in Y

Proof: The directo implication is easy and left as an exercise and it does not