require that X be first countable at x. For the other way implication, Suppose $\{V_1, V_2, \dots, V_n, \dots\}$ is a countable local base at x. We let $W_1 = V_1$, $W_2 = V_1 \cap V_2$, $W_3 = V_1 \cap V_2 \cap V_3$ ..., $W_n = V_1 \cap V_2 \cap \dots \cap V_n$, ... etc. Then the collection $\{W_n : n = 1, 2, \dots\}$ is also a local base at x. Its advantage over the given base is that it is a nested local base, that is, $W_m \subset W_n$ if $m \geq n$. The reason for switching to W's from V's will be clear in the course of the proof.

In order to show that f is continuous at x, we use of one of the characterisation of continuity at a point as that can be deduced from Theorem 8.3. Suppose f is not continuous at x. Then $\exists$ a subset A of X such that $x \in \bar{A}$ but $f(x) \notin \overline{f(A)}$. Now for each n, W_n is a neighbourhood of x and since $x \in \bar{A}$ $W_n \cap A \neq \phi$. Choose $x_n \in W_n \cap A$, for $n \in \mathbb{N}$. Note that for $m \geq n$, $x_m \in W_n$ since $W_m \subset W_n$. We claim that the sequence $\{x_n\}$ converges to x in X. For, let G be an open set containing x. Then since $\{W_1, W_2, \dots, W_n, \dots\}$ is a local base at x,

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there is some n such that, $W_n \subset G$. But then for all $m \geq n$ $x_m \in W_n$ and so $x_m \in G$. So, $\{x_n\}$ converges to x in X. However, for each n, $f(x_n) \in f(A) \subset \overline{f(A)}$. We are assuming that $f(x) \notin \overline{f(A)}$. Then ~~f(x)~~ $Y - \overline{f(A)}$ is an open set which contains $f(x)$, but contains no term of the sequence $\{f(x_n)\}$. Consequently, $\{f(x_n)\}$ does not converge to $f(x)$ in Y. This contradicts the hypothesis and shows that f is continuous at x.

9.10 Definition: Let X be a topological space. ~~A sequence $\{x_n\}$~~ A point x in X is said to be a cluster point of a sequence $\{x_n\}$ in X, if for each neighbourhood N_x of x and if for $n_0 \in \mathbb{N}$, $\exists$ $m \in \mathbb{N}$ such that $m \geq n_0$ and $x_m \in N_x$.

9.11. Theorem Let $A \subset X$, where ~~A~~ X is a first countable space. Then $p \in \overline{A}$ if and only if $\exists$ a sequence $\{x_n\}$ in A converging to p.

Proof: Necessity is easily proved. For proving sufficiency, let $p \in \overline{A}$ and let $\{B_1, B_2, \ldots, B_n, \ldots\}$

be a countable neighbourhood base at p. Without loss of generality, we may take $B_1 \supset B_2 \supset B_3 \ldots \supset B_n \supset \cdots$. Since $p \in \overline{A}$, for each n $\exists$ $x_n \in B_n \cap A$. Clearly, $\{x_n\}$ converges to p.

9.12 Theorem: In a first countable space X, p is a cluster point of a sequence $\{x_n\}$ in X if and only if $\exists$ a subsequence of $\{x_n\}$ converging to p.

Proof: Let $\{B_1, B_2, \ldots\}$ be a countable neighbourhood base at p with $B_1 \supset B_2 \supset B_3 \cdots$. Suppose p is a cluster point of $\{x_n\}$. Then for each $k \in \mathbb{N}$,

$\{x_k, x_{k+1}, \ldots\} \cap B_i \neq \phi$ for each $i = 1, 2, \ldots$

Let n_1 be the first integer in $\mathbb{N}$ such that $x_{n_1} \in B_1$. Assuming that $n_1, n_2, \ldots, n_j$ has been chosen, let n_{j+1} be the first integer greater than n_j such that $x_{n_{j+1}} \in B_{j+1}$. Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ converging to p.

The converse is easy.

Now we introduce the idea of connected topological space.

10. CONNECTED SPACES

10.1 Definition: A topological space X is said to be connected if it is ~~a~~ impossible to find non-empty subsets A and B of it such that

$X = A \cup B$ and $\bar{A} \cap \bar{B} = \phi$.

A space which is not connected is called disconnected. Before giving examples of connected and disconnected spaces it is convenient to reformulate the definition.

10.2 Theorem: Let X be a topological space and A, B subsets of X. Then the following statements are equivalent:

1. $A \cup B = X$ and $\bar{A} \cap \bar{B} = \phi$
2. $A \cup B = X$, $A \cap B = \phi$ and A and B are both closed in X
3. $B = X - A$ and A is clopen (i.e., closed as well as open) in X.
4. $B = X - A$ and ∂A (that is, the boundary of A) is empty.
5. $A \cup B = X$, $A \cap B = \phi$ and A and B are both open in X.

Proof: ~~Clearly~~ (1) $\Rightarrow$ (2): Clearly $\bar{A} \cap \bar{B} = \phi$ implies that $A \cap B = \phi$ since $A \subset \bar{A}$ and $B \subset \bar{B}$. Also $\bar{A} \subset X - \bar{B} \subset X - B = A$ and so $\bar{A} = A$ showing that