A is closed. Similarly, B is closed.

(2) ⇒ (3) is immediate since the complement of a closed set is open.

(3) ⇒ (4): This follows from that fact that boundary of a clopen set is empty (prove it).

(4) ⇒ (5): This requires the converse, that is, a set with empty boundary is clopen (prove it). Also if A is closed then its complement is open.

(5) ⇒ (1): Assume $X = A \cup B$ where $A \cap B = \phi$ and A, B are open in X. Then $A = X - B$ and $B = X - A$. So, A, B are closed sets as well. So, $\bar{A} = A$ and $\bar{B} = B$, showing $\bar{A} \cap \bar{B} = \phi$.

10.3 Theorem: Let X be a topological space. Then the following are equivalent:

1. X is connected
2. X can not be written as the disjoint union of two non-empty closed subsets.
3. The only clopen subsets of X are ϕ and X.
4. Every non-empty proper subset of X has a

non-empty boundary.

5. X Can not be written as disjoint union of two non-empty open subsets.

Proof: The result is immediate from the definition and the last theorem.

From the definition we see that every ~~indis~~ indiscrete space is connected and the only connected discrete spaces are those which consists of at most one point. The space of rational numbers is disconnected; given any irrational number α the sets $\{x \in \mathbb{Q} : x < \alpha\}$ and $\{x \in \mathbb{Q} : x > \alpha\}$ are both open in the relative topology on $\mathbb{Q}$ and $\mathbb{Q}$ is clearly their disjoint union. Similarly the set of irrational numbers is disconnected.

It is clear that if a set is connected w.r.t a ~~topological space~~ topology $\mathcal{T}$ on it, then it is connected w.r.t every topology weaker than $\mathcal{T}$. The following theorem shows that connectedness is preserved under continuous function

10.4 Theorem Let $f: X \to Y$ be continuous surjection. Then if X is connected, so is Y.

Proof: Suppose Y is not connected. Then we can write $Y = A \cup B$ where A and B are disjoint, non-empty and open subsets of Y. But then $X = f^{-1}(A) \cup f^{-1}(B)$. The sets $f^{-1}(A)$ and $f^{-1}(B)$ are mutually disjoint and open since f is continuous. Further each is non-empty as f is a surjection. This contradicts that X is connected. Hence Y is connected.

The first non-trivial examples of connected spaces come from the real line. The proof uses the completeness of real numbers. We shall assume it in the order form, that is, every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound and every non-empty subset of $\mathbb{R}$ which is bounded below has a greatest lower bound.

10.5 Theorem: A subset of $\mathbb{R}$ is connected if and only if it is an interval.

Proof: First note that a subset $X \subset \mathbb{R}$ is an interval if and only if it has the property that for any $a, b \in X$, $(a, b) \subset X$. Now if X is not an interval, $\exists$ real ~~number~~ numbers a, b, c such that $a < c < b$, $a, b \in X$ but $c \notin X$. Let $A = \{x \in X : x < c\}$ and $B = \{x \in X : x > c\}$. Clearly, A, B are disjoint open subsets of X (in the relative topology) since

$A = X \cap (-\infty, c)$ and $B = (c, \infty) \cap X$ and $A \cup B = X$

Further $a \in A$ and $b \in B$, hence A and B are non-empty. This shows that X is not connected.

So, we proved that if X be connected then it is an interval.

Conversely, Suppose X is an interval. Let X be not not connected. So, $X = A \cup B$ where $\bar{A} \cap \bar{B} = \phi$, $A \neq \phi$, $B \neq \phi$ where the closure is relative to X. Let $a_0 \in A$, $b_0 \in B$. Without loss of generality, we may assume that $a_0 < b_0$. Let $x = \frac{a_0 + b_0}{2}$. Then $x \in X$. So x is exactly in one of the sets A and B. If $x \in A$, we rename it as a_1 and rename b_0 as b_1. If $x \in B$, we rename a_0 as a_1 and x as b_1. In any case $[a_1, b_1]$ is an interval with its left end point in A and right end point is in B. We can now take the mid point of $[a_1, b_1]$ and get an interval $[a_2, b_2]$ of half the length with $a_2 \in A$ and $b_2 \in B$. Repeating the process, we get a sequence of intervals $\{[a_n, b_n] : n = 0, 1, 2, 3, \dots\}$ such that $a_n \in A$ and $b_n \in B$