$\forall n \in \mathbb{N}$. Note that $\{a_n\}$ is a bounded monotonically increasing sequence while $\{b_n\}$ is a bounded monotonically decreasing sequence and that $(b_n - a_n) \to 0$ as $n \to \infty$. By the order completeness of $\mathbb{R}$, both sequences converge to a common limit, say, c. Note that $c \in X$, since $a_0 \leq c \leq b_0$. Also every neighbourhood of c intersects A as well as B. So, $c \in \bar{A} \cap \bar{B}$, a contradiction. Hence X is connected.

16.6 Definition: Two subsets A and B of a topological space X are said to be separated if $\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$. Here the closure is w.r.t the space X. The condition is a little stronger than say that A and B are mutually disjoint. But it is weaker than saying that their closures are disjoint. For example, in $\mathbb{R}$, the intervals $(-1, 0)$ and $(0, 1)$ are separated although their closures have 0 as a common point. Note that A and B are separated if and only if they are disjoint closed subsets of $A \cup B$ with relative topology. Note also that a topological space is connected if and only if it is not the union of two non-empty separated ~~sets~~ subsets.

10.7 Theorem : Let X be a topological space and C be a connected subset of X (that is, C with the relative topology is a connected space). Suppose $C \subset A \cup B$ where A, B are separated subsets of X. Then either $C \subset A$ or $C \subset B$.

Proof: Let $G = C \cap A$ and $H = C \cap B$. Then G, H are closed subsets of C since, A, B are closed in $A \cup B$. Also $G \cap H = \phi$. But C is connected. So, either $G = \phi$ or $H = \phi$. In the first case $C \subset B$ while in the second; $C \subset A$.

10.8 Theorem: Let $\mathcal{C}$ be a collection of connected subsets of a topological space X, such that no two members of $\mathcal{C}$ are separated. Then $\bigcup\limits_{C \in \mathcal{C}} C$ is also connected.

Proof: Let $M = \bigcup\limits_{C \in \mathcal{C}} C$. If M is not connected, we could write M as a $A \cup B$ where A, B are non-empty and separated subsets of X. By Theorem 10.7, for each $C \in \mathcal{C}$ either $C \subset A$ or $C \subset B$

we claim that the same possibility holds for all $c \in \mathcal{C}$, i.e., either $C \subset A$ for all $C \in \mathcal{C}$ or $C \subset B$ for all $C \in \mathcal{C}$. If this is not the case, then $\exists$ $C, D \in \mathcal{C}$ such $C \subset A$ and $D \subset B$. But A, B are separated and hence ~~this~~ their subsets C, D are also seperated, contradicting the hypothesis. Thus all member of $\mathcal{C}$ are contained in A or all members of $\mathcal{C}$ are contained in B. Accordingly $M = A$ or $M = B$, contradicting that A, B are both non-empty. So, M is connected.

For applications, it is convenient to have the following corollary of the last theorem:

10.9 Corollary: Let $\mathcal{C}$ be a collection of connected subsets of a ~~of~~ topological space X and suppose K is a connected subset of X (not necessarily a member of $\mathcal{C}$) such that $C \cap K \neq \phi$ $\forall C \in \mathcal{C}$. Then $\left(\bigcup_{C \in \mathcal{C}} C\right) \cup K$ is connected.

Proof: Let $M = \left(\bigcup_{C \in \mathcal{C}} C\right) \cup K$. Let $\mathcal{D} = \{K \cup C : C \in \mathcal{C}\}$. Clearly, $M = \bigcup_{D \in \mathcal{D}} D$. By the theorem 10.8, each member of $\mathcal{D}$ is connected since it is a union of two connected sets which intersect (and which are therefore not separated). Now any two members of $\mathcal{D}$ have at least points of K in common and so are not separated. So again by theorem 10.8, M is connected.

As an application of this corollary, we show that the topological product of two connected spaces is connected.

10.10 Theorem. Let X_1, X_2 be two connected topological spaces. Then $X = X_1 \times X_2$ with the product topology is connected.

Proof: If either X_1 or X_2 is empty then so is X and the result holds trivially. So assume X_1, X_2 are both non-empty. Fix a point $y_1 \in X_1$. Then the subset $\{y_1\} \times X_2$ is homeomorphic to X_2 (prove it) and hence is connected (prove it). Call it K. For each $x \in X_2$, the set $X_1 \times \{x\}$ is similarly connected and its intersection with K is non-empty. Also note that $X_1 \times X_2 = \left(\bigcup_{x \in X_2} X_1 \times \{x\}\right) \cup K$. So by