Corollary 10.9, $X_1 \times X_2$ is connected.

10.11 Corollary: The topological product of any finite number of connected topological space is connected

Proof: If $X_1, X_2, \ldots, X_{n-1}, X_n$ are topological spaces (with $n \geqslant 2$), then $X_1 \times X_2 \times \cdots \times X_n$ is homeomorphic to $(X_1 \times X_2 \times \cdots X_{n-1}) \times X_n$ (Prove it). The result follows by induction on n and the last theorem.

10.12 Theorem: The closure of a connected subset is connected. More generally, if C is a connected subset of a topological space X then any set D such that $C \subset D \subset \bar{C}$ is connected.

Proof: Suppose C is connected and $C \subset D \subset \bar{C}$. If D is not connected then we can write $D = A \cup B$ where A, B are non-empty disjoint and closed relative to D. Then $C \cap A$ and $C \cap B$ are disjoint closed subsets of C whose union is C. But C is connected. So one of them, say, $C \cap B$ is empty. This means $C \subset A$ and hence $\bar{C}^D \subset A$ where the closure is w.r.t. D. But $\bar{C}^D = \bar{C}^X \cap D = D$ since $D \subset \bar{C}^X$. Hence $A = D$, contradicting that B is nonempty. So, D is connected.

10.13 Definition: A component of a topological space is a maximally connected subset, that is, a connected subset which is not properly contained in any connected subset of that space.

For example, the topological space $(-1,0) \cup (0,1)$ with the usual topology has two components $(-1,0)$ and $(0,1)$. A space is connected if and only if it has only one component, namely, the whole space itself. On the other hand in a discrete space the only connected subsets are the empty set and the singleton subsets and hence all components are singleton sets. Such a space is said to be totally disconnected. Discrete spaces are not the only examples of totally disconnected spaces. The set of rationals, the set of irrationals are also totally disconnected. The real line with semi-open interval topology is another example.

Properties of components are summarised in the following:

10.14 Theorem: (a) Components are closed sets (b) Any two distinct components are mutually disjoint. (c) Every non-empty connected subset is contained in a unique component. (d) Every space is the disjoint union of its components.

Proof: (a) Let C be a component of a topological space X.

Then C is connected. Hence by Theorem 10.12, $\bar{C}$ is also connected.
Now $C \subset \bar{C}$. But C is maximal w.r.t the property of
being connected, that is, no proper ~~subset~~ superset of
C can be connected. Hence $C = \bar{C}$ and so C is closed.
(b) Let C, C' be two components. If $C \cap C'$ is
nonempty then by Theorem 10.8, $C \cup C'$ would be connected.
But $C \subset C \cup C'$ and $C' \subset C \cup C'$. So, again, by maximality
of C, C' we get $C = C \cup C' = C'$. Thus two distinct
components are disjoint. (c) Let A be a nonempty
connected subset of X. Let $\mathscr{C}$ be the collection
of all connected subsets of X containing A and let
$M = \bigcup_{c \in \mathscr{C}} c$. Then any two members of $\mathscr{C}$ intersect
and so by Theorem 10.8, M is connected. Clearly
$A \subset M$. We claim M is a component. For, suppose
N is a connected subset of X containing M. Then
$N \in \mathscr{C}$ and so $N \subset M$, Hence $M = N$. In other
words, M is a maximally connected subset of X.
Thus every non-empty connected subset is contained
in a component. Such a component is unique since
two distinct components are disjoint. (d) This
assertion follows from the fact that for any $x \in X$,
$\{x\}$ is a connected set and hence there is

a unique Component C of X such that $x \in C$.

11 Compactness

11.1 Definition: A subset A of a topological space X is said to be a compact subset of X if every cover of A by open subsets of X has a finite subcover.

A space X is said to be compact if X is a compact subset of itself.

11.2. Theorem: Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then A is a compact subset of X If and only if the subspace $(A, \mathcal{T}_A)$ is compact.

Proof: Suppose first that A is a compact subset of X. Let $\mathcal{G}$ be an open cover of the space $(A, \mathcal{T}_A)$. Each member G of $\mathcal{G}$ is of the form $H \cap A$ for some $H \in \mathcal{T}$. For each $G \in \mathcal{G}$, fix $D(G) \in \mathcal{T}$ such that $G = D(G) \cap A$. Then the family $\{D(G) : G \in \mathcal{G}\}$ is a cover of A by open subsets of X. Since A is a compact subset of X, this cover has a finite subcover, say $\{D(G_i) : i = 1, 2, \dots, n\}$ where $G_i \in \mathcal{G}$, $i = 1, 2, \dots, n$. Clearly then $\{G_1, G_2, \dots, G_n\}$ is a finite subcover of $\mathcal{G}$. This