shows that the subspace $(A, \mathcal{T}_A)$ is compact.
Conversely, suppose the subspace $(A, \mathcal{T}_A)$ is compact.
Let $\mathcal{G}$ be a cover of A by open subsets of X.
Then $(G \cap A : G \in \mathcal{G})$ is an open cover of the space $(A, \mathcal{T}_A)$. By compactness of the subspace $(A, \mathcal{T}_A)$, this cover has a finite subcover, say $\{G_i \cap A : i=1,2,\ldots,n\}$ where $G_i \in \mathcal{G}$ for $i=1,2,\ldots,n$. Clearly then, $\{G_1, G_2, \ldots, G_n\}$ is a finite subfamily of $\mathcal{G}$ covering the set A. Thus A is a compact subset of X.

11.3 Every continuous real valued function on a compact space is bounded and attains its extrema.

Proof: Let X be a compact space and suppose $f : X \to \mathbb{R}$ is continuous. First we show that f is bounded. For each $x \in X$, let J_x be the open interval $(f(x)-1, f(x)+1)$ and let $V_x = f^{-1}(J_x)$. By continuity of f, V_x is an open set containing x. Note that f is bounded on each V_x. Now the family $\{V_x : x \in X\}$ is an open cover of X and by compactness, admits a finite subcover say $\{V_{x_1}, V_{x_2}, \ldots, V_{x_n}\}$. Let $M = \max\{f(x_1), f(x_2), \ldots, f(x_n)\} + 1$ and let $m = \min\{f(x_1), f(x_2), \ldots, f(x_n)\} - 1$. Now for any $x \in X$ $\exists$ some i such that $x \in V_{x_i}$. Then $f(x_i) - 1 < f(x) < f(x_i) + 1$, and so $m < f(x) < M$ showing that f is bounded. It remains to show that f attains its bounds. Let L, λ be respectively the supremum and infimum of f over X. ~~such that~~ If there is no point x in X for which $f(x) = L$, then we define a new function $g: X \to \mathbb{R}$ by $g(x) = \dfrac{1}{L - f(x)}$ $\forall x \in X$. Then g is continuous since f is so. However g is unbounded, for given any $R > 0$ $\exists$ x such that $f(x) > L - \frac{1}{R}$ and hence $g(x) > R$. This contradicts the earlier part of the theorem and shows that f attains the value L. Similarly f attains the infimum λ.

11.4 Theorem: Let X be a ~~compact~~ compact space and suppose $f: X \to Y$ is continuous and onto. Then Y is compact. In other words, every continuous image of a compact space is compact.

Proof: Let $\mathcal{V}$ be an open cover of Y. Let $\mathcal{U} = \{f^{-1}(V) : V \in \mathcal{V}\}$. The $\mathcal{U}$ is a cover of X and since f is continuous, it is an open cover of X. Since X is compact, some finite many members of $\mathcal{U}$, say $f^{-1}(V_1), f^{-1}(V_2), \ldots, f^{-1}(V_n)$ where $V_1, V_2, \ldots, V_n \in \mathcal{V}$ cover X. But then $V_1, V_2, \ldots V_n$ cover Y since f is onto. So, Y is compact.

11.5 Theorem: Every closed and bounded interval is compact in $\mathbb{R}$ with usual topology.

Proof: Since any closed and bounded interval $[a,b]$ (with $a<b$) is homeomorphic to the unit interval $[0,1]$, it suffices to prove that $[0,1]$ is compact. Let $\mathcal{U}$ be an open cover of $[0,1]$. An element U of $\mathcal{U}$ is open relative to $[0,1]$ and hence is of the form $V \cap [0,1]$ where V is an open subset of $\mathbb{R}$. Replacing such U's by the corresponding V's, we get a cover $\mathcal{V}$ of $[0,1]$ by sets which are open in $\mathbb{R}$. and evidently it suffices to show that $\mathcal{V}$ has a finite subcover. Now let $S = \{t \in [0,1] :$ the interval $[0,t]$ can be covered by finitely many members of $\mathcal{V}\}$. We have to show that $1 \in S$. Evidently $0 \in S$. So, $S \neq \emptyset$. We claim S is both open and closed in $[0,1]$. First, let $t \in S$. Then $[0,t]$ can be covered by say $V_1, V_2, \ldots, V_n \in \mathcal{V}$ with $t \in V_n$ (say). Since V_n is open, $\exists\ \delta > 0$ such that $(t-\delta, t+\delta) \subset V_n$. Now for any $t' \in (t-\delta, t+\delta) \cap [0,1]$ the interval $[0,t']$ is also covered by $V_1, V_2, \ldots, V_n$ showing $t' \in S$. Hence $(t-\delta, t+\delta) \cap [0,1] \subset S$. Hence S is open, being a neighbourhood of each of its points.

On the other hand, suppose $t \in [0,1] - S$. choose $V \in \mathcal{V}$ such that $t \in V$. V being open $\exists\ \delta > 0$ such that $(t-\delta, t+\delta) \subset V$. We claim that $(t-\delta, t+\delta) \cap [0,1] \subset [0,1] - S$. For, let $t' \in (t-\delta, t+\delta) \cap [0,1]$. If $t' \in S$ then $[0,t']$ can be covered by, say, $V_1, V_2, \ldots, V_n \in \mathcal{V}$. But then $V_1, V_2, \ldots, V_n$ and V ~~cover~~ together cover $[0,t]$ contradicting $t \notin S$. Thus, we have shown that $[0,1] - S$ is also open. Putting it together, S is a non-empty clopen subset of $[0,1]$. but we know that $[0,1]$ ~~by Theorem 10.5~~ is connected (by Theorem 10.5). We are thus forced to conclude that S is the entire interval $[0,1]$ and hence, in particular $1 \in S$ as was to be proved.

11-6 Theorem: Let X be a Hausdorff space, $x \in X$ and F a compact subset of X not containing x. Then $\exists$ open sets U, V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$.

Proof: For each $y \in F$, $\exists$ open sets U_y, V_y such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \phi$. The family $\{V_y : y \in F\}$ is an open cover of F. Since F is compact, $\exists$ a finite subcover, say, $\{V_{y_1}, V_{y_2}, \ldots, V_{y_n}\}$. Let $U = \bigcap_{i=1}^{n} U_{y_i}$ and $V = \bigcup_{i=1}^{n} V_{y_i}$. Then U, V are disjoint