complements of members of $\mathcal{F}$ is a topology on X in which the family of closed sets is just $\mathcal{F}$

1.5. Basis for a topology

For each of the examples previously given for topology, we are able to specify the topology by describing the entire collection $\mathcal{T}$ of open sets. Usually, this is too difficult. In most cases one specifies instead a smaller collection of subsets of X and defines the topology in terms of that.

Definition: If X is a non-empty set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that,

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If $x \in B_1 \cap B_2$, B_1, B_2 are basis elements, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Definition: If $\mathcal{B}$ is a basis for a topology on X, the topology $\mathcal{T}$ generated by $\mathcal{B}$ is described as follows: A subset U of X is said to be open in X (that is to be element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$

Note that each element of $\mathcal{B}$ is open in X under this definition, so that $\mathcal{B} \subset \mathcal{T}$. It is easy to check that this collection of subsets of X is a topology on X. But first let us consider some examples.

__Example 1__ Let $\mathcal{B}$ be the collection of all circular regions (interiors of circles) in the plane. Then $\mathcal{B}$ satisfies both conditions for a basis. The second condition is illustrated in Figure 1. In the ~~top~~ topology generated by $\mathcal{B}$, a subset U ~~is~~ of the plane is open if every x in U lies in some circular region contained in U.

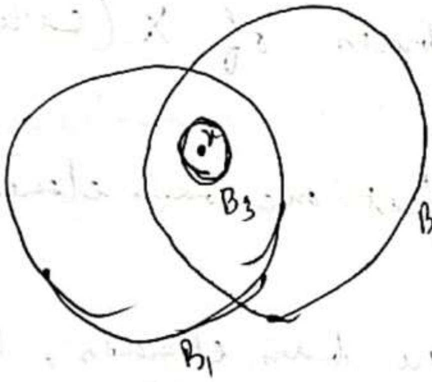

Figure 1

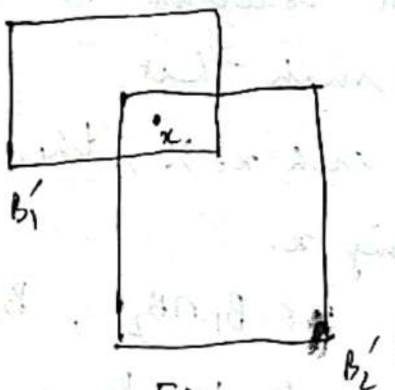

Figure 2

__Example 2__ Let $\mathcal{B}'$ be the collection of all rectangular regions (interiors of rectangles) in the plane, where rectangles have sides parallel to the coordinate axes. Then $\mathcal{B}'$ satisfies both the condition for a basis. The second condition is illustrated in Figure 2; in this case the condition is trivial, because the intersection of any two basis elements is itself a basis element (or empty). As we shall see later, the basis $\mathcal{B}'$ generates the same topology on the plane as the basis $\mathcal{B}$ given in the preceeding example.

Example 3 If X is any non-empty set, the collection of all one-point subsets of X is a basis for the discrete topology on X.

Let us now check that the collection $\mathcal{T}$ generated by the basis $\mathcal{B}$ is, in fact, a topology on X. If U is the empty set, it satisfies the defining condition of openness vacuously. Likewise, $X \in \mathcal{T}$, since for each $x \in X$ there is some basis element B containing x and contained in X. Now let us take an indexed family $\{U_\alpha : \alpha \in I\}$ of elements of $\mathcal{T}$, and show that $U = \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$. Given $x \in U$, there is an index α such that $x \in U_\alpha$. Since U_α is open, there is a basis element B such that $x \in B \subset U_\alpha$. Then $x \in B$ and $B \subset U$, so that U is open, by definition. Now let us take $U_1, U_2 \in \mathcal{T}$ and show that $U_1 \cap U_2 \in \mathcal{T}$. Given $x \in U_1 \cap U_2$, choose a basis element B_1 containing x such that $B_1 \subset U_1$; choose also a basis element B_2 containing x such that $B_2 \subset U_2$. The second condition for a basis enables us to choose a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Then $x \in B_3$, and $B_3 \subset U_1 \cap U_2$. So, $U_1 \cap U_2 \in \mathcal{T}$, by definition. By induction, we can show that for $U_1, U_2, \ldots, U_n \in \mathcal{T}$, $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$.

<u>Theorem 1.6</u>: Let X be a non-empty set; let $\mathcal{B}$ be a basis for a topology ~~of~~ $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Proof: Given a collection of elements of $\mathcal{B}$, they are also elements of $\mathcal{T}$. Because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$. So, U equals ~~as~~ a union of elements of $\mathcal{B}$.

<u>Theorem 1.7</u> Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively on X. Then the following are equivalent:

(1) $\mathcal{T}'$ is finer that $\mathcal{T}$.

(2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$

Proof: (2) $\Rightarrow$ (1). Let $U \in \mathcal{T}$, we ~~are~~ want to show $U \in \mathcal{T}'$. Let $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is $B \in \mathcal{B}$ such that $x \in B \subset U$. Condition (2) tells us there exists