open subsets, $x \in U$ and $F \subset V$

11.7 Corollary: A compact subset in a Hausdorff space is closed.

Proof: Suppose X is a T_2 space and F is a compact subset of X. Then by Theorem 11.6, for any $x \in X-F$ $\exists$ open sets U, V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$. In particular, $U \cap F = \phi$ and hence $U \subset X-F$. Thus $X-F$ is a neighbourhood of each of its points. So $X-F$ is open and F closed.

11.8 Corollary: Every continuous function from a compact into a T_2 space is closed.

Proof: Suppose $f: X \to Y$ is continuous where X is compact and Y is Hausdorff. Let C be a closed subset of X. Then C is compact (prove it). But then $f(C)$ is closed in Y by Corollary 11.7. Hence images of closed sets in X are closed in Y, i.e., the map f is closed.

11.9 Corollary: A continuous bijection from a compact space onto a Hausdorff space is a homeomorphism.

Proof: Let $f: X \to Y$ be a continuous bijection where X is compact and Y is Hausdorff. We claim f is open. Let G be an open subset of X.

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Then $X-G$ is closed and hence $f(X-G)$ is closed in Y by Corollary above. But $f(X-G) = Y - f(G)$ because f is a bijection. So, $f(G)$ is open in Y. Thus f is continuous, open bijection and hence a homeomorphism.

11.9 The Concepts of Compactness in a metric space and sequentially compactness of a metric space.

A metric space (X, d) is also a topological space where the collection of open sets on X forms a topology on X. We have defined the concept of compactness in a metric space on Page-66 of CC13 Notes and sequentially compactness of a metric space in Page-69 of CC13 Notes.

11.9.1 Definition: A metric space (X, d) is said to have Bolzano-Weierstrass Property (BWP) if every infinite subset of X has a limit point.

11.9.2 Theorem: A metric space is sequentially compact if and only if it has the BWP.

Proof: Let (X, d) be a metric space. Assume first that X is sequentially compact. Let A be an infinite subset of X. We can extract a sequence $\{x_n\}$ of distinct points from A as A is infinite.

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Since X is sequentially compact, $\{x_n\}$ has a subsequence, say $\{x_{n_k}\}$ which converges to a point x in X.

claim : x is a limit point of A.

Let $\varepsilon > 0$ be given. Since $x_{n_k} \to x$ as $k \to \infty$, $\exists$ a positive integer N such that

$$\cancel{x_{n_k} \in B_\varepsilon(x)} \; x_{n_k} \in B_\varepsilon(x) \quad \forall\, k \geq N$$

But $x_{n_k} \neq x$, $\forall k \geq N$, because $\{x_n\}$ consists of distinct elements from A. Thus $B_\varepsilon(x)$ contains an element x_{n_k}, for some $k \geq N$, of A, different from x. Hence x is a limit point of A.

Thus every infinite subset of X has a limit point in X and hence X has the BWP.

Conversely, assume that X has the BWP; in other words, every infinite subset of X has a limit point in X. Let $\{x_n\}$ be an arbitrary sequence in X. In order to prove the result, we need to establish that $\{x_n\}$ has a convergent subsequence. Let A be the range of the sequence $\{x_n\}$.

Case (i) A is a finite set

This is possible only when at least one of the elements appears infinitely many times in the sequence. Then we get a constant subsequence (consisting of that element repeatedly infinitely many times), hence a convergent subsequence.

Case (ii) A is an infinite set. Since the space X has the BWP and A is an infinite subset of X, it has at least one limit point, say, x in X. Then $B_1(x)$ must have an infinitely many elements from A. Let us choose one such element, say, x_{n_1} other than x. Again $B_{1/2}(x)$ must have infinitely many elements of A and hence choose x_{n_2}, $n_2 > n_1$. Similarly $B_{1/3}(x)$ must contain infinitely many elements of A and hence choose x_{n_3}, $n_3 > n_2$, and so on. This way we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$x_{n_k} \in B_{1/k}(x), \quad \forall k$$

$$\Rightarrow d(x_{n_k}, x) < \frac{1}{k} \to 0 \text{ as } k \to \infty$$

$$\Rightarrow x_{n_k} \to x$$

Hence every sequence $\{x_n\}$ has a convergent subsequence and X is sequentially compact.

THIS IS THE END OF DSEB (Point Set Topology) Notes.

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