an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$. Then $x \in B' \subset U$.

So, $U \in \mathcal{T}'$ by definition.

(1) $\Rightarrow$ (2) we are given $x \in X$ and $B \in \mathcal{B}$ with $x \in B$.

Now $B \in \mathcal{T}$ by definition and $\mathcal{T} \subset \mathcal{T}'$ by condition (1); therefore $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, there is an element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Note: One now understand (through by figure) that the collection of $\mathcal{B}$ of all circular regions in the plane generates the same topology as the collection $\mathcal{B}'$ of all rectangular regions is shown by Figure 3.

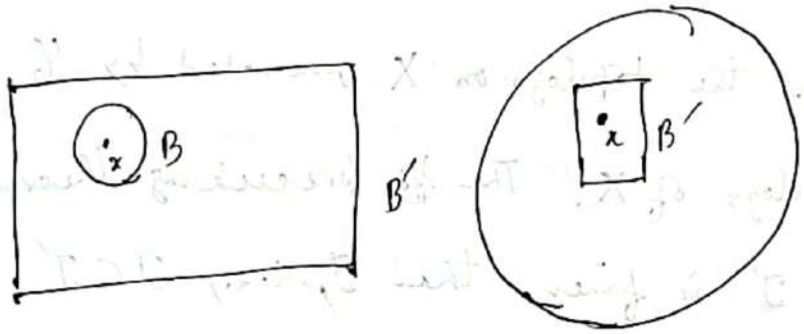

Figure 3

We have described earlier ways how to go from a basis to the topology it generates. Sometimes we need to go in the reverse direction, from a topology to a basis generating it. Here is one way of obtaining a basis for a given topology; we shall use it frequently.

Theorem 1.8 Let X be a topological space. Suppose that $\mathcal{C}$ be a collection of open sets of X such that for each open set U of X and each $x \in U$, $\exists$ an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Proof: We must show that $\mathcal{C}$ is a basis. The first condition for a basis is easy: Given $x \in X$, since X is ~~open set~~ itself ~~open~~ an open set there is by ~~the~~ hypothesis an element $C \in \mathcal{C}$ such that $x \in C \subset X$. To check the second condition, let $x \in C_1 \cap C_2$, where $C_1, C_2 \in \mathcal{C}$. Since C_1 and C_2 are open, so is $C_1 \cap C_2$. So, $\exists$ by hypothesis an element $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$.

Let $\mathcal{T}'$ denotes the topology on X generated by $\mathcal{C}$; let $\mathcal{T}$ be the topology of X. The ~~B~~ preceeding theorem shows that $\mathcal{T}'$ is finer than $\mathcal{T}$, i.e., $\mathcal{T} \subset \mathcal{T}'$. ~~Conv~~ Conversely, since element of $\mathcal{C}$ is an element of $\mathcal{T}$, so are arbitrary unions of elements of $\mathcal{C}$. So, by Theorem 1.6, $\mathcal{T}' \subset \mathcal{T}$. So, we conclude $\mathcal{T} = \mathcal{T}'$.

There are two interesting topologies on the line ~~R~~ $\mathbb{R}$ which can be described in terms of bases:

Definition: If $\mathcal{B}$ is the collection of all open intervals in the real line $(a,b) = \{x : a < x < b\}$, the topology generated by $\mathcal{B}$ is called the standard topology on the real line.

If $\mathcal{B}'$ is the collection of all half-open intervals of the form $[a, b) = \{x : a \leq x < b\}$, where $a < b$, the topology generated by $\mathcal{B}'$ is called the lower limit topology on $\mathbb{R}$. When $\mathbb{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_\ell$.

It is easy to see that both $\mathcal{B}$ and $\mathcal{B}'$ are bases; the intersection of two basis elements either another basis element or is empty.

Whenever we consider $\mathbb{R}$, we shall suppose that it is given the standard topology unless we specifically state otherwise. The lower limit topology will prove very useful to us in constructing counterexamples. The relation between these two topologies is the following:

<u>Theorem 1.9</u> The lower limit topology $\mathcal{T}'$ on $\mathbb{R}$ is strictly finer than the standard topology $\mathcal{T}$.

<u>Proof:</u> Given a basis element (a, b) for $\mathcal{T}$, and a point $x \in (a, b)$, the basis element $[x, b)$ for $\mathcal{T}'$ contains x and lies in (a, b). Therefore, $\mathcal{T}'$ is finer than $\mathcal{T}$. On the other, given a basis element $[x, d)$

for $\mathcal{T}'$, there is no open ~~set~~ interval (a,b) satisfying the condition $x \in (a,b) \subset [x,d)$. ;
then $\mathcal{T}$ is not finer than $\mathcal{T}'$.

A question may occur to you at this point. Since the topology generated by a basis $\mathcal{B}$ may be described as the collection of arbitrary unions of elements of $\mathcal{B}$, what happens if you start with a given collection of sets and take finite intersections of them as well as arbitrary unions? This question leads to the notion of a subbasis for a topology.

Definition : A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose unions equals X. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all union of finite intersections of elements of $\mathcal{S}$.

We must of course check that $\mathcal{T}$ is a topology. For this purpose, it will suffice to show that the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis, for then the collection $\mathcal{T}$ of all unions of elements of $\mathcal{B}$ is a topology, by theorem 1.6. Given $x \in X$, it belongs to an element of $\mathcal{S}$ and