and hence to an element of $\mathcal{B}$; this is the first condition for a basis. To check the second condition, let $B_1 = S_1 \cap \dots \cap S_m$ and $B_2 = S_1' \cap \dots \cap S_n'$ be two elements of $\mathcal{B}$. Their intersection

$$B_1 \cap B_2 = (S_1 \cap \dots \cap S_m) \cap (S_1' \cap \dots \cap S_n')$$

is also a finite intersection of elements of $\mathcal{S}$, so it belongs to $\mathcal{B}$.

2. Neighbourhoods of a point, interior points, limit points, derived set.

Definition 2.1 Let $(X, \mathfrak{T})$ be a topological space, $x_0 \in X$ and $N \subset X$. Then N is said to be a neighbourhood (nbd in short) of x_0 or x_0 is said to be an interior point of N if $\exists$ an open set $V \in \mathfrak{T}$ such that $x_0 \in V \subset N$.

Note that the set N itself is not required to be open.

2.2 (Proposition): A subset of a topological space is open if and only if (iff) it is ~~not~~ a nbd of each of its points.

Proof. Let X be a topological space and $G \subset X$. First suppose G is open. Then evidently G is a nbd of each of its points. Conversely, suppose G is a nbd of each of its points. Then for each $x \in G$, $\exists$ an open set V_x such that $x \in V_x \subset G$. So, $G = \bigcup_{x \in G} V_x$. Since each V_x is open, so is G.

Trivially if N is a nbd of a point x then so is any superset of N. It is also easy to show that the intersection of any two (and hence finitely many) nbds of a point is again a nbd of that point.

Definition 2.3: Let A be a subset of a topological space X and $y \in X$. Then y is said to be a limit point (or an accumulation point) of A if every open set containing y contains at least one point of A other than y.

As examples we note that in a discrete space (i.e space with discrete topology) no point is a limit point of any set while at the other extreme, in an indiscrete space (i.e., space with indiscrete topology) a point y is a limit point of any set A provided only that A contains at least one point besides y. In the usual topology on the real line, every real number is a limit point of the set of rational numbers while the set of integers has no limit point.

Definition 2.4: Let A be a subset of ~~the~~ a topological space X. Then the derived set of A, denoted by A', is the set of all limit points of A in X.

3. Closure of a subset

Definition 3.1 : If X be a topological space and $E \subset X$, the closure of E in X, denoted by $\bar{E}$ or $Cl(E)$, is defined as $\bar{E} = Cl(E) = \bigcap \{K \subset X : K \text{ is closed and } E \subset K\}$

Where confusion is possible as to what space the closure is to be taken, we will write $Cl_X(E)$. As arbitrary intersection of closed sets is closed, $\bar{E}$ is closed. It is the smallest closed set containing E, in the sense that it is contained in every closed set containing E

Lemma 3.2 If $A \subset B$, then $\bar{A} \subset \bar{B}$

Proof: Since $B \subset \bar{B}$, if $A \subset B$, then we have $A \subset \bar{B}$; since $\bar{B}$ is closed, we must then have $\bar{A} \subset \bar{B}$

Theorem 3.3 The operation $A \to \bar{A}$ in a topological space X has the following properties:

K-a) $E \subset \bar{E}$

K-b) $\overline{(\bar{E})} = \bar{E}$

K-c) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

K-d) $\overline{\phi} = \phi$

K-e) E is closed in X iff $\overline{E} = E$

Moreover, given a set X and a mapping $A \to \overline{A}$ of $\mathcal{P}(X)$ into $\mathcal{P}(X)$ satisfying K-a through K-d, if we define closed sets in X using K-e, the result is a topology on X whose closure operation is just the operation $A \to \overline{A}$ we began with. ($\mathcal{P}(X)$ is the power set of X)

<u>Proof</u>: First suppose X is a topological space. We shall show K-c holds, leaving the rest of K-a through K-e as an easy exercise. Since $\overline{A} \cup \overline{B}$ is closed and contains $A \cup B$, it contains $\overline{A \cup B}$. On the other hand, since $A \subset A \cup B$ and $B \subset A \cup B$, we have $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$, by Lemma 3.2, and thus $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

So, $\overline{A \cup B} = \overline{A} \cup \overline{B}$. This establishes K-c.

We proceed to the ~~sec~~ second part of the Theorem. Let X be any set and $A \to \overline{A}$ a mapping of $\mathcal{P}(X)$ into $\mathcal{P}(X)$ satisfying K-a through K-d. Let $\mathcal{F}$ be the collection of all sets A such that $\overline{A} = A$. We assert that $\mathcal{F}$ satisfies the three conditions