(i) (i), (ii) and (iii) of Theorem 1.4

First note that if $A \subset B$, then by K-c, $\bar{B} = \bar{A} \cup \overline{(B-A)}$

so that $\bar{A} \subset \bar{B}$

Now suppose $F_\alpha \in \mathcal{F}$ for each $\alpha \in I$. Then since

$\bigcap_{\alpha \in I} F_\alpha$ is contained in F_α, $\overline{\left(\bigcap_{\alpha \in I} F_\alpha\right)} \subset \bar{F}_\alpha$ for each $\alpha \in I$

and hence $\overline{\left(\bigcap_{\alpha \in I} F_\alpha\right)} \subset \bigcap_{\alpha \in I} \bar{F}_\alpha = \bigcap_{\alpha \in I} F_\alpha$. But the

reverse inclusion is given by K-a, so,

$$\overline{\left(\bigcap_{\alpha \in I} F_\alpha\right)} = \bigcap_{\alpha \in I} F_\alpha$$

. Thus (i) of Theorem 1.4 holds.

Next suppose $F_1, F_2, \ldots, F_n \in \mathcal{F}$. Then by K-c and induction $\overline{F_1 \cup F_2 \cup \cdots \cup F_n} = \bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_n = F_1 \cup F_2 \cup \cdots \cup F_n$.

So, $F_1 \cup F_2 \cup \cdots \cup F_n \in \mathcal{F}$. This establishes (ii) of Theorem 1.4

By K-d and K-a, it is clear that ϕ and X, respectively, belong to $\mathcal{F}$, so (iii) of Theorem 1.4 is established.

Thus $\mathcal{F}$ is a collection of closed sets for X.

It remains to show that the resulting closure operation in X is just the operation $A \to \bar{A}$ we began with; that is, that $\bar{A}$ is the smallest element of $\mathcal{F}$ containing A, for each $A \subset X$. Since $\overline{(\overline{A})} = \overline{A}$ by K-b, we know that $\overline{A} \in \mathcal{F}$, and from K-a, we know that $A \subset \overline{A}$. If K is any element of $\mathcal{F}$ containing A, then $\overline{A} \subset \overline{K} = K$. Thus $\overline{A}$ is indeed the smallest element of $\mathcal{F}$ containing A.

An operation $A \to \overline{A}$ in a set X which satisfies K-a through K-d is called Kuratowski closure operation (which, incidentally, is the reason for the letter K in the numeration). Thus every Kuratowski closure operation determines and is determined by some topology.

Examples: 1. Let X be an infinite set and for each $A \subset X$, define $\overline{A}$ as follows:

$$\overline{A} = A \quad \text{if } A \text{ is finite}$$
$$\overline{A} = X \quad \text{if } A \text{ is infinite}$$

The properties K-a through K-d can be verified for the resulting operation $A \to \overline{A}$, so we have a Kuratowski closure operation in X. The resulting topology on X, the cofinite topology, has closed sets those sets A for which $\overline{A} = A$.

Note: We always have $\overline{A \cup B} = \bar{A} \cup \bar{B}$. The corresponding statement for intersections is not true. Let X be $\mathbb{R}$, A the rationals in $\mathbb{R}$, B the irrationals in $\mathbb{R}$, and give X the usual topology. Check that $\bar{A} = \mathbb{R}$ and $\bar{B} = \mathbb{R}$. But $A \cap B = \phi$, so, $\overline{A \cap B} = \phi$. Thus $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

It is always true that $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$

2. The closure of a subset A of a discrete space is A itself.
3. The closure of any non-empty subset of a set X with trivial or indiscrete topology is X (and, of course, the closure of ϕ is ϕ.).

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4. Interior of a subset

Definition 4.1 If X be a topological space and $E \subset X$, the interior of E in X, denoted by E^0 or $\mathrm{Int}(E)$, is defined as $E^0 = \mathrm{Int}(E) = \bigcup \{ G \subset X : G \text{ is open and } G \subset E \}$.

When confusion might otherwise result, we will write $\mathrm{Int}_X(E)$. As arbitrary union of open sets is open, E^0 is open. It is the largest open set contained in E, in the sense that it contains any other open set contained in E.

The notions of interior and closure are dual to each other, in much the same way that "open" and "closed" are. The strictly formal nature of this duality can be brought out in observing that

$$X - E^{\circ} = \overline{X - E}$$

and

$$X - \overline{E} = (X - E)^{\circ}$$

(Exercise)

Thus any theorem about closures in a topological space can be translated to a theorem about interiors. The next two results, are, for example, the dual results to 3.2 and 3.3.

4.2 Lemma: If $A \subset B$, then $A^{\circ} \subset B^{\circ}$

Proof: It is clear that $A^{\circ} \subset A$, so if $A \subset B$, we have $A^{\circ} \subset B$. Thus A° is an open set contained in B. So, $A^{\circ} \subset B^{\circ}$.

Theorem 4.3. The interior operation $A \to A^{\circ}$ in a topological space X has the following properties.

I-a) $A^{\circ} \subset A$

I-b) $(A^{\circ})^{\circ} = A^{\circ}$

I-c) $(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$

I-d) $X^{\circ} = X$

I-e) G is open iff $G^{\circ} = G$.