conversely, given any map $A \to A^\circ$ of $\mathcal{P}(X)$ into $\mathcal{P}(X)$ in a set X, satisfying I-a through I-d, if open sets are ~~def~~ defined in X using I-e, the result is a topology on X in which the interior of a set $A \subset X$ is just A°.

Proof: The proof can be done directly or by using the translation process on 3.2. Either way, it is easy and left as an exercise.

Examples: 1. In $\mathbb{R}$, with the usual topology, the interior of a closed interval $[a,b]$ is (a,b). In $\mathbb{R}^2$ with the usual topology, the interior of the disk

$$\{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$$

is the disk $\{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$

2. In $\mathbb{R}$, with the usual topology, if A is the set of ~~in~~ rationals and B the set of irrationals, then

$$A^\circ = B^\circ = \phi. \text{ But } (A \cup B)^\circ = \mathbb{R}^\circ = \mathbb{R}.$$

Hence $(A \cup B)^\circ \neq A^\circ \cup B^\circ$

But it is always true that $A^\circ \cup B^\circ \subset (A \cup B)^\circ$

5. Boundary of a set

5.1 Definition If X be a topological space and $E \subset X$, the boundary of E or frontier of E, denoted by ~~∂A or $Fr(A)$~~ ∂E or $Fr(E)$, is $\overline{E} \cap \overline{(X-E)}$. Evidently the set $\partial E = Fr(E) = \overline{E} \cap \overline{(X-E)}$.

The boundary of E is a closed set.

We now give the relationship between the boundary, closure and interior operations.

5.2 Theorem For any subset E of a topological space X,

a) $\overline{E} = E \cup \partial E$

b) $E^0 = E - \partial E$

c) $X = E^0 \cup \partial E \cup (X-E)^0$

Proof: a)
$$E \cup \partial E = E \cup (\overline{E} \cap \overline{X-E})$$
$$= (E \cup \overline{E}) \cap (E \cup \overline{X-E})$$
$$= \overline{E} \cap X = \overline{E}$$

b)
$$E - \partial E = E - (\overline{E} \cap \overline{(X-E)})$$
$$= (E - \overline{E}) \cup (E - \overline{X-E})$$
$$= E - \overline{X-E} = E^0$$

c) Since $\partial E \cup (X-E) = \overline{X-E}$ (as is easily verified) and since $X - E^{\circ} = \overline{X-E}$,

We have $X = E^{\circ} \cup \overline{X-E} = E^{\circ} \cup \partial E \cup (X-E)^{\circ}$

Examples: The boundary of the closed interval $[a,b]$ in $\mathbb{R}$ is $\{a,b\}$, as is the boundary of any interval with the same end points. If $\mathbb{Q}$ denotes the set of rationals in $\mathbb{R}$, $\partial\mathbb{Q} = \mathbb{R}$

b) For any space X; $\partial X = \phi$.

c) If D is the closed unit disk in the plane, and $X = \mathbb{R}^2$, $\partial D = S^1 = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$

where $D = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$.

5.3 Theorem For a subset A in a topological space X; $\overline{A} = A \cup A'$, where A' is the derived set of A.

Proof: First we claim that $A \cup A'$ is closed or that $X - (A \cup A')$ is open. We do so by showing that $X - (A \cup A')$ is a nbd of each of its points.** Let $y \in X - (A \cup A')$. Then since y is not a limit point of A $\exists$ an open set V containing y such that V contains no point ~~except~~ of A except possibly y. But $y \notin A$, so we have $A \cap V = \phi$. We claim $A' \cap V = \phi$ also. For, let $z \in A' \cap V$. Then V is an open set containing z which is a limit point of A. So, $V \cap A$ is nonempty, a contradiction. So $A' \cap V = \phi$ and hence $V \subset X - (A \cup A')$. This proves that $A \cup A'$ is closed and since it obviously contains A, it also contains $\bar{A}$, i.e., $\bar{A} \subset A \cup A'$.

For the other way inclusion, $A \cup A' \subset \bar{A}$, it suffices to show that $A' \subset \bar{A}$. Since we already have $A \subset \bar{A}$. So, let $y \in A'$. If $y \notin \bar{A}$, then $y \in X - \bar{A}$ which is an open set since $\bar{A}$ is always a closed set. But y is a limit point of A. So, $(X - \bar{A}) \cap A \neq \phi$ which is a contradiction since $X - \bar{A} \subset X - A$.

So, $y \in \bar{A}$. This completes the proof.

Now let X be a topological space and for $x \in X$,