let $\mathcal{N}_x$ be the set of all nbds of x in X (w.r.t the given topology on X). The family $\mathcal{N}_x$ is called the neighbourhood system at x. In the following theorem we list some properties of $\mathcal{N}_x$

5.4 Theorem The nbd system $\mathcal{N}_x$ at x in a topological space X has the following properties:

N-a) If $U \in \mathcal{N}_x$, then $x \in U$,

N-b) If $U, V \in \mathcal{N}_x$, then $U \cap V \in \mathcal{N}_x$,

N-c) If $U \in \mathcal{N}_x$, then $\exists$ a $V \in \mathcal{N}_x$ such that

$U \in \mathcal{N}_y$ for each $y \in V$,

N-d) If $U \in \mathcal{N}_x$ and $U \subset V$, then $V \in \mathcal{N}_x$ and furthermore

N-e) $G \subset X$ is open iff G contains a nbd of each of its points (** This result is used to prove $X - (A \cup A')$ is open in previous Theorem 5.3, so this result should be established earlier)

Conversely, if in a set X, a collection $\mathcal{N}_x$ of subsets of X is assigned to each $x \in X$ so as to satisfy N-a through N-d, if if N-e is used to define "open", the result is a topology in X; in which the nbd system at $x \in X$ is precisely $\mathcal{N}_x$.

Proof: N-a is obvious. For N-b, if $U, V \in \mathcal{N}_x$, then $x \in U^0$ and $x \in V^0$, so $x \in U^0 \cap V^0 = (U \cap V)^0$

and hence $U \cap V \in \mathcal{N}_x$. For N-c, let $U \in \mathcal{N}_x$ and pick $V = U^\circ$. Then for each $y \in V$, $y \in U^\circ$, so $U \in \mathcal{N}_y$.
For N-d, if $U \in \mathcal{N}_x$, then $x \in U^\circ$. If $U \subset V$, then $U^\circ \subset V^\circ$, so $x \in V^\circ$. Hence $V \in \mathcal{N}_x$.
Finally, to prove N-e, if G is open, then $G = G^\circ$ and G is a nbd of each of its points. On the other hand, if each $x \in G$ has a nbd $V_x \subset G$, then $G = \bigcup_{x \in G} V_x^\circ$ is a union of open sets and thus open.

The converse assertion is left as an exercise.

5.5 Theorem For a subset A of a topological space X,
$\bar{A} = \{y \in X :$ every nbd of y meets A non-vacuously$\}$
i.e., $\{y \in X : U \in \mathcal{N}_y \Rightarrow U \cap A \neq \phi\}$.

Proof: Let $B = \{y \in X : U \in \mathcal{N}_y \Rightarrow U \cap A \neq \phi\}$. We have to show that $\bar{A} = B$. By Theorem 5.3, this amounts to showing that $A \cup A' = B$.
First let $y \in A \cup A'$. If $y \in A$ then certainly every nbd of y meets A at least at the point y and so $y \in B$. If $y \in A'$ then too, by the definition of a limit point, every nbd of

y contains a point of A and so $y \in B$.

Thus $A \cup A' \subset B$. Conversely, let $y \in B$. If $y \notin A \cup A'$, then $y \notin \bar{A}$ and so $X - \bar{A}$ is a nbd of y which does not meet A, contradicting that $y \in B$. So $B \subset A \cup A'$. Thus $B = \bar{A}$

5.6 Definition Let A be a subset of a topological space X. Then A is said to be dense in X if $\bar{A} = X$.

5.7 Theorem A subset A of a topological space X is dense in X iff for every non-empty open set subset B of X, $A \cap B \neq \phi$.

Proof: Suppose A is dense in X and B is a non-empty open set in X. If $A \cap B = \phi$ then $A \subset X - B$ whence $\bar{A} \subset X - B$ since $X - B$ is closed. But then $X - B \subsetneqq X$ contradicting that $\bar{A} = X$. So, $A \cap B \neq \phi$. Conversely, assume that A meets every non-empty open subsets of X. This clearly means that the only closed set containing A is X and consequently $\bar{A} = X$

It is now easy to show that in the real line with usual topology the set Q of all rational numbers, as

well as its complement $\mathbb{R}-\mathbb{Q}$ are both dense in $\mathbb{R}$.
For a topological space X, the set X is always dense in X.

6. Subspaces

A subset of a topological space inherits a topology of its own, in an obvious way. This topology and some of its easily developed properties will be given here.

6.1 Definition: If (X, τ) is a topological space and $A \subset X$, the collection $\tau' = \{G \cap A : G \in \tau\}$ is a topology for A, called the relative topology for A. The fact that a subset of X is being given this topology is signified by referring to it as a subspace of X.

Any time a topology is used on a subset of a topological space without explicitly being described, it is assumed to be the relative topology. This natural and convenient convention has the result that any adjective which can be applied to spaces (e.g., "compact") can be applied automatically to subsets of a topological space. But be careful that we are not saying that if a space has a particular property, then every subspace of that space has the same property.