6.2 Examples: a) The real line, regarded as the x-axis in $\mathbb{R}^2$, inherits its usual topology from $\mathbb{R}^2$. The set of integers $\mathbb{Z}$, as a subspace of $\mathbb{R}$ inherits the discrete topology.

b) Any subspace of a discrete space is discrete and any subspace of trivial or indiscrete space is trivial or indiscrete.

[Note: The topology τ' is denoted by τ_A or τ/A]

c) A subspace of a subspace is a subspace. i.e For a topological space (X, τ), and for $A_1 \subset A_2$, $(\tau_{A_2})_{A_1} = \tau_{A_1}$. The proof is left as an easy exercise.

The open sets in a subspace A of X are the intersections with A of the open sets in X. Most, but not all, of the related topological notions are introduced in the same way, by intersection, as the following theorem and example show.

6.3 Theorem If A is a subspace of a topological space (X, τ), then

(a) $H \subset A$ is open in A iff $H = G \cap A$, where G is open in X.

(b) $F \subset A$ is closed iff $F = K \cap A$ where K is closed in X.

(c) if $E \subset A$, then $Cl_A E = A \cap Cl_X E$

(d) if $x \in A$, then V is a nbd of x in A iff

$V = U \cap A$, where U is a nbd of x in X.

(e) If $\mathcal{B}$ is a base for X, then

$\{B \cap A : B \in \mathcal{B}\}$ is a base for A

Proof: (a) is just the definition of the subspace topology on A, recorded here for reference.

(b) follows directly from (a)

(c) follows from (b) and the definition of closure of E as the intersection of all closed sets containing E.

(d) follows from (a) and the definition of a nbd of x as a set containing an open set containing x.

(e) ~~Ex~~ Proof is an exercise.

The reader will notice that two concepts are missing from the list above; no mention ~~of~~ is made of the interior operator ~~and~~ or the boundary operator in ~~of~~ subspaces. The following examples indicate why this is so.

6.4 **Examples** (a) Let X be the plane with usual topology while $A = E =$ the x-axis

Then $\text{int}_A E = A$ while $\text{int}_X E = \phi$, so that the former cannot be obtained by intersecting the latter with A. It is always true, however, that

$$\text{int}_A E \supset A \cap \text{Int}_X E$$

(b) Using the example, we have $\partial_A E = \phi$ while $\partial_X E = A$, so that, again, the former can not be obtained by intersecting the latter with A. It is always true, however, that $\partial_A E \subset A \cap \partial_X E$.

Note: Let A be a subspace of a topological space X. If U is open in A and A is open in X, then U is open in X.

Proof: easy exercise.

7. The product topology on $X \times Y$

If X and Y are topological spaces, there is a standard way of defining a topology on the cartesian product $X \times Y$. We consider this topology now and study some of its properties.

7.1 Definition: Let X and Y be topological spaces. The product topology on $X \times Y$ is the having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Let us check that $\mathcal{B}$ is a basis. The first condition is trivial, since $X \times Y$ is itself a basis element. The second condition is ~~also~~ almost as easy, since the intersection of two basis elements $U_1 \times V_1$ and $U_2 \times V_2$ is another basis element because

$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$ and the latter set is a basis element because $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y respectively. Note that the collection $\mathcal{B}$ is not a topology on $X \times Y$. The union of the two basis elements may not be a basis element, but it i