open in $X \times Y$.

Each time we introduce a new concept, we shall try to relate it to the concepts introduced earlier. In the present case, we ask: What can one say if the topologies in X and Y are given by bases? The answer is as follows:

__Theorem 7.2__ If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a ~~topology~~ basis for the topology of Y, then the collection $\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$

Proof: We apply Theorem 1.8. Given an open set W of $X \times Y$ and a point $(x, y) \in W$, by definition of the product topology $\exists$ a basis element $U \times V$ such that $(x, y) \in U \times V \subset W$. Because $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y respectively, we can choose an element B of $\mathcal{B}$ such that $x \in B \subset U$ and $\exists$ an element C of $\mathcal{C}$ such that $y \in C \subset V$. Then $(x, y) \in B \times C \subset U \times V$. Thus the collection $\mathcal{D}$ meets the criterion of Theorem

1.8. So, $\mathcal{D}$ is a basis for $X \times Y$.

Example 1 We consider the set of all real numbers $\mathbb{R}$, with standard topology. The product of this topology with itself is called the standard topology on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. It has as basis the collection of all products of open sets of $\mathbb{R}$, but the Theorem just proved tells us that the much smaller collection of all products $(a, b) \times (c, d)$ of open intervals in $\mathbb{R}$ will also serve as a basis for the topology of $\mathbb{R}^2$

It is sometimes useful to express the product topology in terms of a subbasis. To do this we first define certain functions called projections.

Definition: Let $\pi_1 : X \times Y \to X$ be defined by the equation $\pi_1(x, y) = x$;

let $\pi_2 : X \times Y \to Y$ be defined by the equation

$$\pi_2(x, y) = y$$

The maps π_1 and π_2 are called the projections of $X \times Y$ onto its first and second factors, respectively.

We use the word 'onto' because π_1 and π_2 are surjective (unless the one of the spaces X and Y to be empty, in which case $X \times Y$ is empty and our whole discussion is empty as well!).

If U is open subset of X, then the set $\pi_1^{-1}(U)$ is precisely the set $U \times Y$, which is open in $X \times Y$. Similarly, if V is open in Y, then $\pi_2^{-1}(V) = X \times V$, which is also open in $X \times Y$. The intersection of these two sets is the set $U \times V$. This fact leads to the following theorem:

Theorem 7.3 The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof: Let $\mathcal{T}$ denote the product topology on $X \times Y$; let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Because every element of $\mathcal{S}$ belongs to $\mathcal{T}$, so do arbitrary unions of finite intersections of elements of $\mathcal{S}$. Thus $\mathcal{T}' \subset \mathcal{T}$. On the other hand, every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite intersection of elements of $\mathcal{S}$, since

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V).$$

Therefore, $U \times V$ belongs to $\mathcal{T}'$, so that $\mathcal{T} \subset \mathcal{T}'$ as well.

We have defined a topology on the product $X \times Y$ of two topological spaces. Now we generalize this definition to arbitrary cartesian products. There are two ways of generalizing the definition; the one that will later prove to be the more important we shall call the product topology.

One way to impose a topology on a product space is the following; it is a direct generalization of the way ~~as~~ we defined a basis for the product topology on $X \times Y$;

<u>Definition</u>: Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take a basis for a topology on the product space $\prod_{\alpha \in J} X_\alpha$ the collection of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where U_α is open in X_α for each $\alpha \in J$. The topology generated by this basis is called the box topology.

This collection satisfies the first condition for a