Notes on my portion of GE I (MTMG)

SB (Subhasundar Bandyopadhyay)
Deptt. of Mathematics, GGGDC

Books that are to be followed :

1. Comprehensive Degree Mathematics - D. Chatterjee & B.K. Pal (Vol. 1 & Vol. 2)
2. Higher Algebra - J.G. Chakravorty & P.R. Ghosh
3. Analytical Geometry - J.G. Chakravorty & P.R. Ghosh.
4. Differential Calculus - K.C. Maity & R.K. Ghosh

The portion of the syllabus we start with is :

Rank of a matrix ; Determination of rank either by considering minors or by sweep-out process. Consistency and solution of a system of linear equations with not more than 3 variables by matrix method.

1. Rank of a matrix :

We first define submatrices and minors of a matrix.

Any matrix obtained by taking some rows and columns of a given $m\times n$ matrix A is called a ~~submatrix~~ sub-matrix of A.

Examples : $\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ is a sub-matrix of $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 7 & 1 \\ 5 & 4 & 7 \end{bmatrix}$ by taking first row and third row and second column and third column

$[3, 7, 1]$ is a sub-matrix taking second row and first, second and third column.

$\begin{bmatrix} 3 & 7 & 1 \\ 5 & 4 & 7 \end{bmatrix}$ is a sub-matrix taking second and third row and first, second and third column. So, a submatrix may be a square matrix or may not be a square matrix.

The determinant of ~~the~~ a square submatrix of order r, obtained from a given $m\times n$ matrix A by taking r rows and r columns, is called a minor of A of order r.

So, we now find some minors of a particular matrix, Consider the matrix $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 1 & 3 & 7 \\ 2 & 5 & 7 & 3 \end{bmatrix}$.

Highest order minor of A that can be form, is of order minimum of 3 and 4 = 3 as A has 3 rows and 4 columns. A minor of order 3 is the determinant

$\begin{vmatrix} 2 & 1 & 4 \\ 4 & 1 & 7 \\ 2 & 5 & 3 \end{vmatrix}$ by ~~taking~~ forming a ~~submatrix~~ sub-matrix

by taking 3 rows and first, second and fourth column and taking its determinant.

value of this minor $= 2\times(3-35) - 1\times(12-14) + 4\times(20-2)$
$= -64+2+72 = 10$

How many minors of order 3 can be formed from A?

The answer is $^3c_3 \times {^4c_3}$ (choosing 3 rows from 3 rows and simultaneously choosing 3 columns from 4 columns).
$= \frac{4\times3\times2}{3\times2\times1} = 4$

They are $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 1 & 3 \\ 2 & 5 & 7 \end{vmatrix}$, $\begin{vmatrix} 2 & 1 & 4 \\ 4 & 1 & 7 \\ 2 & 5 & 3 \end{vmatrix}$, $\begin{vmatrix} 2 & 3 & 4 \\ 4 & 3 & 7 \\ 2 & 7 & 3 \end{vmatrix}$ and $\begin{vmatrix} 1 & 3 & 4 \\ 1 & 3 & 7 \\ 5 & 7 & 3 \end{vmatrix}$

A minor of order 2 is $\begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} = 20-2 = 18$. It is formed by taking the determinant of the submatrix formed by taking second and third row and first and second column.

How many minors of order 2 can be formed from A?

The answer is $^3c_2 \times {^4c_2} = \frac{3\times2}{2\times1} \times \frac{4\times3}{2\times1} = 18$ (choosing 2 rows from 3 rows and simultaneously choosing 2 columns from 4 columns)

Some of them are $\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}$, $\begin{vmatrix} 3 & 7 \\ 7 & 3 \end{vmatrix}$, $\begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix}$ etc.

Scanned with CamScanner

A minor of order 1 is 2, formed by taking the determinant of the submatrix formed by taking only first row and first column. So, each element of the matrix is a minor of order 1.

How many minors of order 1 can be formed?

The answer is the number of elements of the matrix $= 4 \times 3 = {}^4c_1 \times {}^3c_1 = 12$

If a minor has value which is non-zero, it is called a non-zero minor.

<u>Definition of rank of a matrix</u>: Let A be a non-zero matrix of order $m \times n$. The rank of A is defined to be the greatest positive integer r such that A has at least one non-zero minor of order r. The rank of a zero matrix is defined to be 0.

<u>Examples</u>: 1. Let $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 1 & 3 & 7 \\ 2 & 5 & 7 & 3 \end{bmatrix}$. We want to find the rank of A

The highest order minor of A is of order 3.

Now $\begin{vmatrix} 2 & 1 & 4 \\ 4 & 1 & 7 \\ 2 & 5 & 3 \end{vmatrix} = 10 \neq 0$ So, rank of $A = 3$.

2. Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 5 \\ 2 & 0 & 6 \end{bmatrix}$

The highest order minor of A is of order 3 and the only minor of order 3 $= \det A = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 5 \\ 2 & 0 & 6 \end{vmatrix} = 0$

($\det A$ = determinant of A)

So, rank of $A < 3$ and since there is a second order minor $\begin{vmatrix} 1 & 0 \\ 4 & -1 \end{vmatrix} = -1 \neq 0$, rank of $A = 2$

Scanned with CamScanner

3. Let $A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 3 & 0 & 4 & 2 \\ 6 & 9 & -3 & 3 \end{bmatrix}$

The highest order minor of A is of order 3 and they are $^4C_3 = \frac{4\times3\times2}{3\times2\times1} = 4$ in numbers. They are

$$\begin{vmatrix} 2 & 3 & -1 \\ 3 & 0 & 4 \\ 6 & 9 & -3 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 3 & 1 \\ 3 & 0 & 2 \\ 6 & 9 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & -1 & 1 \\ 3 & 4 & 2 \\ 6 & -3 & 3 \end{vmatrix} = 0 \text{ and}$$

$$\begin{vmatrix} 3 & -1 & 1 \\ 0 & 4 & 2 \\ 9 & -3 & 3 \end{vmatrix} = 0$$

So, every minor of order 3 is zero and so rank of A is less than 3. There is a second order minor $\begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} = -9 \neq 0$ and so rank of $A = 2$.

Home Work: HW1. Determine the rank of the following matrices:

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 3 & 5 & -1 \\ 2 & 1 & -2 & 8 \\ 0 & 5 & 12 & -10 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 4 & -1 & 2 \\ 2 & 8 & -2 & 4 \\ -1 & -4 & 1 & -2 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 2 & 0 & 1 \\ 3 & -4 & -4 & -7 \\ -7 & 5 & 6 & 10 \end{bmatrix}$

HW2. Find the value of a if the rank of the matrix $\begin{bmatrix} 1 & a & 2 \\ 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ is 2.

Scanned with CamScanner