The nth power function and its inverse

Case 1 Let n be an even positive integer and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n$, $x \in \mathbb{R}$

Then f is not injective on $\mathbb{R}$

Let $I = [0, \infty)$ and let $f : I \to \mathbb{R}$ be defined by $f(x) = x^n$, $x \in I$. Then f is continuous, one-to-one and strictly increasing on I and $f(I) = [0, \infty)$. The range of f is $[0, \infty)$

Hence there exists an inverse function $g : [0, \infty) \to [0, \infty)$ such that g is continuous and strictly increasing on $[0, \infty)$. g is defined by $g(y) = \sqrt[n]{y}$, $y \in [0, \infty)$

$$g \circ f(x) = g(f(x)) = g(x^n) = \sqrt[n]{x^n} = x$$

$$f \circ g(y) = f(g(y)) = f(\sqrt[n]{y}) = (\sqrt[n]{y})^n = y$$

g is called the nth root function (n even) and it is defined on $[0, \infty)$

Case 2 Let n be an odd positive integer

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n$, $x \in \mathbb{R}$.

Then f is continuous and strictly increasing on $\mathbb{R}$.

The range of f is $\mathbb{R}$. Hence there exists an inverse function $g : \mathbb{R} \to \mathbb{R}$ such that g is continuous and strictly increasing on $\mathbb{R}$.

g is defined by $g(y) = \sqrt[n]{y}$, $y \in \mathbb{R}$.

g is called the nth root function (n odd) and it is defined on $\mathbb{R}$.

Sine function and its inverse

Let $f: [-\pi/2, \pi/2] \to \mathbb{R}$ be defined by $f(x) = \sin x$, $x \in [-\pi/2, \pi/2]$. Then f is strictly increasing and continuous on $[-\pi/2, \pi/2]$

The range of f is $[-1, 1]$

Hence there exists an inverse function $g: [-1,1] \to [-\pi/2, \pi/2]$ such that g is continuous and strictly increasing on $[-1,1]$. g is defined by $g(y) = \sin^{-1} y$, $y \in [-1,1]$

g is called the principal inverse sine function.

Similarly, we can define all the principal inverse trigonometric functions.

Derivative - its geometrical and physical interpretation

Let $I = [a,b]$ be an interval and $f: I \to \mathbb{R}$ be a real valued function on I. (i) Let $c \in [a,b]$ such that $a < c < b$

Then f is said to have derivatives at c or f is differential at c if $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists or

$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists (putting $x - c = h$)

So, f has derivative at c if

$\lim_{h \to 0+} \frac{f(c+h) - f(c)}{h}$ exists and $\lim_{h \to 0-} \frac{f(c+h) - f(c)}{h}$ exists

and they are equal. If the limit be l, l is said to be derivative at c and denoted by $f'(c)$

(ii) Let c be the left end point a. Then f is said to be differentiable at a if

$\lim_{h \to 0+} \frac{f(a+h)-f(a)}{h}$ exists. If $\lim_{h \to 0+} \frac{f(a+h)-f(a)}{h} = l$

then l is called derivative of f at a and denoted by $f'(a)$

(iii) Let c be the right end point b.

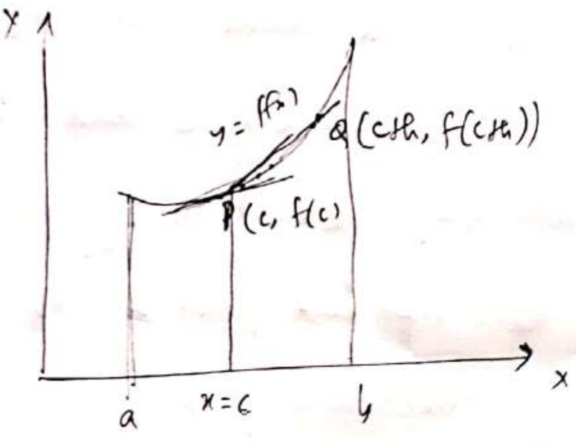
f is said to have derivative at b if

$\lim_{x \to h-} \frac{f(b+h)-f(b)}{h}$ exists. If l be the limit, l is called the derivative of f at b and is denoted by $f'(b)$

<u>Physical interpretation</u>: At $x=c$, the value of $f(x)$ is $f(c)$ and at $c+h$, the value of f is $f(c+h)$.

So, for a change $c+h-c=h$ of x, the value ~~changes~~ of $f(x)$ changes $f(c+h)-f(x)$. So, the average rate of change is $\frac{f(c+h)-f(c)}{h}$. When $h \to 0$ $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$ is instantaneous rate of change at c.

<u>Geometrical interpretation</u>



Let $y=f(x)$ be curve curve drawn for the function $f:[a,b] \to \mathbb{R}$. Let $P(c, f(c))$ be the point on the curve at $x=c$ and let Q be a neighbouring point on the curve whose coordinates are $(c+h, f(c+h))$

Now slope of the chord PQ is $\frac{f(c+h) - f(c)}{c+h-c} = \frac{f(c+h) - f(c)}{h}$

If $Q \to P$ then PQ becomes the tangent at P and $h \to 0$

and the slope of the tangent at P is $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

So, if $f'(c)$ exists then it is the slope of the tangent at $(c, f(c))$

Sign of the derivative

Let $I = [a,b] \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a real valued function on I.

Let $c \in I$ and $a < c < b$

Then f is said to be increasing at c if there exist positive number h such that $f(x) < f(c)$ for all $x \in I$ satisfying $c-h < x < c$ and $f(x) > f(c)$ for all $x \in I$ satisfying $c < x < c+h$.

f is said to be decreasing at c if there exists a positive number h such that $f(x) > f(c)$ for all $x \in I$ satisfying $c-h < x < c$ and $f(x) < f(c)$ for all $x \in I$ satisfying $c < x < c+h$.

Let c be the left end point a of I.

f is said to increasing at a if there exists a positive number h such that $f(x) > f(a)$ for all $x \in I$ satisfying $a < x < a+h$

f is said to be decreasing at a if there exists a positive number h such that $f(x) < f(a)$ for all $x \in I$ satisfying $a < x < a+h$