Let c be the right end point b of I.

f is said to increasing at b if there exists a positive h such that $f(x) < f(c)$ for all $x \in I$ satisfying $c-h < x < c$.

f is said to be decreasing at b if there exists a positive h such That $f(x) > f(c)$ for all $x \in I$ satisfying $c-h < x < c$

We state a theorem without proof

Theorem Let $I \subset \mathbb{R}$ be an interval and a function $f: I \to \mathbb{R}$ be differentiable at $c \in I$.

(i) If $f'(c) > 0$, then f is increasing at c

(ii) If $f'(c) < 0$, then f is decreasing at c.

Note 1. A function f may be increasing (or decreasing) at a point c in its domain of definition without being differentiable at c.

For example, the function f defined by

$$f(x) = \begin{cases} x, & x < 1 \\ 2x-1, & x \geq 1 \end{cases}$$

is increasing at 1 but f is not differentiable at 1

The function f defined by

$$f(x) = \begin{cases} 1-x, & x < 0 \\ 1-2x, & x \geq 0 \end{cases}$$

is decreasing at 0 but f is not differentiable at 0.

Note 2 If f is increasing at a point c, then $f'(c)$ may not be positive. For example, let $f(x) = x^3$, $x \in \mathbb{R}$. f is increasing at 0, but $f'(0) = 0$

If f is decreasing at a point c then $f'(c)$ may not be negative. For example, let $f(x) = -x^3$, $x \in \mathbb{R}$, $\therefore f$ is decreasing at $x = 0$ but $f'(0) = 0$

Differential - application in finding approximation

Definition If the derivative exists, the

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$ defining the derivative

is equivalent to the equation

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \epsilon$$

where $\epsilon \to 0$ as $h \to 0$. From this, we obtain

$$f(x+h) - f(x) = h f'(x) + \epsilon h \quad \ldots \quad (1)$$

In this relation, consider x to be fixed and h to be variable. The relation (1) gives the increment of the function, namely $f(x+h) - f(x)$. Thus, when derivative exists, the increment of the function consists of two parts - one part is $h f'(x)$, called the principal part or linear part and the other part, ϵh, called error. The principal part is proportional to h and the error can be made as small as we like by making h small enough. $f(x+h) - f(x)$ is written as Δy. Thus when $h \to 0$, the error part becomes more and more insignificant compared with the principal part. We call the principal part $h f'(x)$ as the differential of the function, denoted by $d f(x)$

If $y = f(x)$, we write the differential of the function

$$= df(x) = dy = hf'(x) = h\frac{dy}{dx} \quad \ldots (2)$$

In particular, if $f(x) = x$ then $f'(x) = 1$ and (2) becomes $dx = h$

So that we may write

$$dy = df(x) = f'(x)\,dx = \left(\frac{dy}{dx}\right) dx$$

Example 1 Obtain dx, dy, Δy, $\Delta y - dy$, given

$$y = \frac{x^2}{2} + 3x, \quad x = 2 \quad \text{and} \quad \Delta x = 0.5$$

Solution: $dx = \Delta x = 0.5$, $\Delta y = \left\{\frac{(2.5)^2}{2} + 3 \times 2.5\right\} - \left(\frac{2^2}{2} + 3 \times 2\right)$
$= 2.625$

$$dy = \frac{dy}{dx}dx = (x+3)\,dx = (2+3) \times 0.5 = 2.5$$

Hence $\Delta y - dy = 2.625 - 2.5 = 0.125$

Example 2 Find approximately the volume of a spherical shell of external diameter 10 cm and thickness $\frac{1}{16}$ cm

Solution. Since $V = \frac{4}{3}\pi r^3$ (r = radius, V = volume)

The exact volume is ΔV = Volume of the sphere of radius 5 cm − volume of the sphere of radius $4\frac{15}{16}$ cm.

So, we require an approximate value of ΔV, we proceed to find dV, $dV = \frac{dV}{dr} \cdot dr = 4\pi r^2 dr$

Substitute $r = 5$ and $dr = -\frac{1}{16}$, we obtain $dV = -19.625$ cc . So, $|dV| = 19.625$ cc

Observe that exact volume $\Delta V = 19.4$ cc.

Small errors It is occasionally convenient to approximate small errors by use of differentials.

Small errors in the value of the function arise due to a small error in the independent variable.

Example 3 A right circular cone has its altitude equal to the radius of the base. The altitude is measured as 10 cm with a possible error in the measurement of 0.01 cm. Find approximately the greatest possible error in the calculated volume which this possible error may produce.

Solution: Given $h = r$ and we know $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi h^3$ (V = Volume, r = radius of the base, h = altitude) --- (1)

The exact maximum error in V ~~is~~ will be the increment ΔV in its ~~value~~ value obtained from (1) when h ~~too~~ changes from 10 cm to 10.01 cm. The approximate error is the value of ΔV. Hence $dV = \pi h^2 dh = \pi \times 10^2 \times 0.01 = \pi$ c.c

This means the calculated volume $1000\pi/3$ cc will differ by π cc approximately; the computed value may be too great or too small as we substitute $dh = \pm 0.01$

Example 4 Given that $\log_{10} e = 0.4343$, find $\log_{10} 10.1$.

Solution $d(\log_{10} x) = d(\log_e x \times \log_{10} e) = \frac{1}{x} \log_{10} e \cdot dx$

Put $x = 10$, $\log_{10} e = 0.4343$, $dx = 0.1$

then $d(\log_{10} x) = \frac{0.4343 \times 0.1}{10} = 0.004343$

So, $\log_{10} 10.1 = \log_{10} 10 + 0.004343 = 1.004343$