Successive derivative : Let I be an interval and a function $f: I \to \mathbb{R}$ be differentiable at a point $c \in I$. If f be differentiable at every point in some ~~neighbourhood~~ subinterval $I_1(c)$ such that $c \in I_1(c) \subset I$, then $f': I_1(c) \to \mathbb{R}$ is a function on $I_1(c)$. If f' is differentiable at c then the derivative of f' at c is called the second order derivative of f at c and is denoted by $f''(c)$. This is to note that c may also be an end point of the sub-interval $I_1(c)$. If f' be differentiable at every point of some subinterval $I_2(c)$ such that $c \in I_2(c) \subset I_1(c)$, then $f'': I_2(c) \to \mathbb{R}$ is a function on $I_2(c)$. If f'' is differentiable at c then the derivative of f'' at c is called the Third order derivative at c and is denoted by $f'''(c)$.

In a similar manner, we define the nth order derivative $f^{(n)}(c)$ when the derivative exists.

This is to emphasize that in order that the nth derivative of f may exist at c, f^{n-1} must be defined on subinterval containing c, allowing the possibility of c to be an end point also of such sub-intervals.

Note: If we write $y = f(x)$ then $f'(x) = \frac{dy}{dx} = y_1$ is also written.

So, we write $f^n(x) = \frac{d^n y}{dx^n} = y_n$

Now we find some nth order derivatives of some functions.

1. Let $f(x) = x^{\alpha}$, $\alpha \in \mathbb{R}$

Then $f'(x) = \alpha x^{\alpha-1}$

$f''(x) = \alpha(\alpha-1)x^{\alpha-2}$

$f'''(x) = \alpha(\alpha-1)(\alpha-2)x^{\alpha-3}$

$\cdots \qquad \cdots$

$f^{n}(x) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\,x^{\alpha-n}$

Note: Here the result has been obtained by inference. But it can be proved by principle of mathematical induction.

Corollary: Let $f(x) = x^{m}$, m is a positive integer

Then $f'(x) = m x^{m-1}$

$f''(x) = m(m-1)x^{m-2}$

$f'''(x) = m(m-1)(m-2)x^{m-3}$

$\cdots \qquad \cdots$

$f^{m}(x) = m!$

and $f^{m+r}(x) = 0$ for $r = 1, 2, \ldots$

2. Let $f(x) = \dfrac{1}{ax+b}$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, $ax+b \neq 0$

Then $f'(x) = (-1)(ax+b)^{-2} \cdot a$

$f''(x) = (-1)(-2)(ax+b)^{-3} \cdot a^2$

$\cdots \qquad \cdots$

$f^{n}(x) = (-1)^{n} n!\,(ax+b)^{-n-1} \cdot a^{n}$

3. Let $f(x) = e^{ax+b}$, $a \in \mathbb{R}$, $b \in \mathbb{R}$

Then $f'(x) = e^{ax+b} \cdot a$

$$f''(x) = e^{ax+b} \cdot a^2$$

$\cdots \quad \cdots$

$$f^n(x) = e^{ax+b} \cdot a^n$$

4. Let $f(x) = \log(ax+b)$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, $ax+b > 0$

Then $f'(x) = (ax+b)^{-1} \cdot a$

$$f''(x) = (-1)(ax+b)^{-2} \cdot a^2$$

$$f'''(x) = (-1)(-2)(ax+b)^{-3} \cdot a^3$$

$\cdots \quad \cdots$

$$f^n(x) = (-1)^{n-1}(n-1)!\,(ax+b)^{-n} \cdot a^n$$

5. Let $f(x) = \sin ax$, $a \in \mathbb{R}$

Then $f'(x) = \cos ax \cdot a = \sin(\pi/2 + ax) \cdot a = a \sin(\pi/2 + ax)$

$$f''(x) = \cancel{a^2\cos(\pi/2 + ax)} - a^2 \sin ax = a^2 \sin(2\pi/2 + ax)$$

$$f'''(x) = -a^3 \cos ax = a^3 \sin(3\pi/2 + ax)$$

$\cdots \quad \cdots \quad \cdots$

$$f^n(x) = a^n \sin\left(\frac{n\pi}{2} + ax\right)$$

6. If $f(x) = \cos ax$, $a \in \mathbb{R}$ Then $f^n(x) = a^n \cos\left(n\frac{\pi}{2} + ax\right)$

7. Let $f(x) = e^{ax} \sin bx$, $a \in \mathbb{R}$, $b \in \mathbb{R}$

Then $f'(x) = e^{ax}(a \sin bx + b \cos bx)$

Let $a = r\cos\theta$, $b = r\sin\theta$, $-\pi < \theta \le \pi$. Then $r^2 = a^2 + b^2$

Then $f'(x) = re^{ax}(\sin bx \cos\theta + \cos bx \sin\theta) = re^{ax} \sin(bx+\theta)$

$$f''(x) = r^2 e^{ax} \sin(bx + 2\theta)$$

$\cdots$

$$f^n(x) = r^n e^{ax} \sin(bx + n\theta), \text{ where } r\cos\theta = a,\ r\sin\theta = b$$

8. If $f(x) = e^{ax}\cos bx$, $a \in \mathbb{R}$, $b \in \mathbb{R}$

Then $f^n(x) = r^n e^{ax}\cos(bx + n\theta)$, where $r\cos\theta = a$, $r\sin\theta = b$.

Leibnitz's theorem: If u and v are two functions of independent variable x, then nth derivative of the product uv is given by

$$(uv)_n = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \cdots {}^nc_r u_{n-r} v_r + \cdots + u v_n$$

where $(uv)_r = \frac{d^r(uv)}{dx^r}$ and $u_r = \frac{d^r u}{dx^r}$, $v_r = \frac{d^r v}{dx^r}$, $r = 1, 2, \cdots, n$

Proof: This Theorem is proved by mathematical induction.

$$(uv)_1 = u_1 v + u v_1$$

So, the theorem is true for $n = 1$

So, assume that the theorem is true for $n = m$

So, $(uv)_m = u_m v + {}^mc_1 u_{m-1} v_1 + {}^mc_2 u_{m-2} v_2 + \cdots + {}^mc_{r-1} u_{m-r+1} v_{r-1}$
$+ {}^mc_r u_{m-r} v_r + \cdots + u v_m$

Differentiating both sides, we have

$$(uv)_{m+1} = u_{m+1} v + u_m v_1 + {}^mc_1 u_m v_1 + {}^mc_1 u_{m-1} v_2 + \cdots + {}^mc_{r-1} u_{m-r+2} v_{r-1} + {}^mc_{r-1} u_{m-r+1} v_r + {}^mc_r u_{m-r+1} v_r + {}^mc_r u_{m-r} v_{r+1} + u_1 v_m + u v_{m+1}$$

$$= u_{m+1} v + (1 + {}^mc_1) u_m v_1 + ({}^mc_1 + {}^mc_2) u_{m-1} v_2 + \cdots + ({}^mc_{r-1} + {}^mc_r) u_{m-r+1} v_r + \cdots + u v_{m+1}$$

$$= u_{m+1} v + {}^{m+1}c_1 u_m v_1 + {}^{m+1}c_2 u_{m-1} v_2 + \cdots + {}^{m+1}c_r u_{m+1-r} v_r + \cdots + u v_{m+1}$$