So, the formula is true $n = m+1$

Hence by the principle of mathematical induction,

$(uv)_n = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \cdots + {}^nc_r u_{n-r} v_r + \cdots + u v_n$

for any positive integer n.

Worked Examples

1. Find the nth derivative of $\dfrac{x^2+4x+1}{x^3+2x^2-x-2}$

Solution : Let $y = \dfrac{x^2+4x+1}{x^3+2x^2-x-2}$

$$= \frac{x^2+4x+1}{x^2(x+2)-(x+2)}$$

$$= \frac{x^2+4x+1}{(x^2-1)(x+2)}$$

$$= \frac{x^2+4x+1}{(x-1)(x+1)(x+2)}$$

Applying partial fractions,

$$y = \frac{x^2+4x+1}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2} \quad \text{(say)}$$

Then $x^2+4x+1 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$

Putting $x = 1, -1, -2$ respectively, we obtain $A = 1$, $B = 1$ and $C = -1$

So, $y = \cancel{\dfrac{1}{(x-1)} + \dfrac{1}{(x+1)}} \; \dfrac{1}{x-1} + \dfrac{1}{x+1} - \dfrac{1}{x+2}$

Now applying nth derivative of $\dfrac{1}{ax+b}$, we have

$$y_n = (-1)^n n! \left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}} - \frac{1}{(x+2)^{n+1}} \right]$$

2. Find y_n, if $y = x\log\frac{x-1}{x+1}$

Solution: Given $y = x\log\frac{x-1}{x+1}$

$$= x\log(x-1) - x\log(x+1)$$

So, $$y_1 = \frac{x}{x-1} + \log(x-1) - \frac{x}{x+1} - \log(x+1)$$

$$= \log\frac{x-1}{x+1} + \cancel{\frac{2x}{(x-1)(x+1)}} \; \frac{x}{x-1} - \frac{x}{x+1}$$

$$= \log\frac{x-1}{x+1} + 1 + \frac{1}{x-1} - \left(1 - \frac{1}{x+1}\right)$$

$$= \log\frac{x-1}{x+1} + \frac{1}{x-1} + \frac{1}{x+1}$$

So, $$y_2 = \frac{1}{x-1} - \frac{1}{x+1} - \frac{1}{(x-1)^2} - \frac{1}{(x+1)^2}$$

Applying $(n-2)$th derivative, we have

$$y_n = (-1)^{n-2}\left[\frac{(n-2)!}{(x-1)^{n-1}} - \frac{(n-2)!}{(x+1)^{n-1}} - \frac{(n-1)!}{(x-1)^n} - \frac{(n-1)!}{(x+1)^n}\right]$$

$$= (-1)^{n-2}\left\{\frac{(n-2)!}{(x-1)^n}\left[(x-1) - (n-1)\right] - \frac{(n-2)!}{(x+1)^n}\left[(x+1) + (n-1)\right]\right.$$

$$= (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$$

$$= (-1)^n(-1)^{-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$$

$$= (-1)^n(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$$

3. Find y_n if $y = \sin 4x \cdot \cos 2x$

Solution: Given $y = \sin 4x \cdot \cos 2x$

$$= \frac{1}{2}\left(\sin 6x + \sin 2x\right)$$

So, $$y_n = \frac{1}{2}\left[6^n \sin\left(6x + \frac{n\pi}{2}\right) + 2^n \sin\left(2x + \frac{n\pi}{2}\right)\right]$$

4. Find y_n if $y = \cos^4 x$

Solution: Given $y = \cos^4 x$

$$= (\cos^2 x)^2$$

$$= \left(\frac{1+\cos 2x}{2}\right)^2$$

$$= \frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8}(1 + \cos 4x)$$

$$= \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

So, $$y_n = \frac{1}{2}\,2^n \cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8}\,4^n \cos\left(4x + \frac{n\pi}{2}\right)$$

$$= 2^{n-1}\cos\left(2x + \frac{n\pi}{2}\right) + 2^{2n-3}\cos\left(4x + \frac{n\pi}{2}\right)$$

5. Find the nth derivative of $e^{2x}\cos^2 x \sin x$

Solution: Let $y = e^{2x}\cos^2 x \sin x$

$$= e^{2x}\left(\frac{1+\cos 2x}{2}\right)\sin x$$

$$= \frac{1}{2}e^{2x}\sin x + \frac{1}{2}e^{2x}\cos 2x \sin x$$

$$= \frac{1}{2}e^{2x}\sin x + \frac{1}{2}e^{2x}\cdot\frac{1}{2}(\sin 3x - \sin x)$$

$$= \frac{1}{4}e^{2x}\sin x + \frac{1}{4}e^{2x}\sin 3x$$

Now differentiating n times with respect to x, we have,

$$y_n = \frac{1}{4}(2^2+1)^{n/2} e^{2x} \sin\left(x + n\tan^{-1}\frac{1}{2}\right) + \frac{(2^2+9)^{n/2}}{4} e^{2x}\sin\left(3x+\tan^{-1}\frac{3}{2}\right)$$

$$= \frac{1}{4} 5^{n/2} e^{2x}\sin\left(x + n\tan^{-1}\frac{1}{2}\right) + \frac{(13)^{n/2}}{4} e^{2x}\sin\left(3x + \tan^{-1}\frac{3}{2}\right)$$

6. If $y = x^2 e^x$, find y_n

Solution:

Let $y = x^2 e^x$, $u = e^x$, $v = x^2$

Then by Leibnitz's theorem

$$y_n = x^2(e^x)_n + {}^nC_1(2x)(e^x)_{n-1} + {}^nC_2(2)(e^x)_{n-2}$$

$$= x^2e^x + 2nxe^x + \frac{n(n-1)}{2!}2e^x.$$

$$= x^2e^x + 2nxe^x + (n^2-n)e^x$$

$$= (x^2 + 2nx + n^2 - n)e^x$$

7. If $y = x^2 \sin x$, find y_n

Solution: Here $y = x^2\sin x$, $u = \sin x$, $v = x^2$

So, by Leibnitz's theorem,

$$y_n = x^2(\sin x)_n + {}^nC_1(2x)(\sin x)_{n-1} + {}^nC_2(2)(\sin x)_{n-2}$$

$$= x^2\sin\left(x+\frac{n\pi}{2}\right) + n\cdot 2x\cdot\sin\left(x+(n-1)\frac{\pi}{2}\right) + n(n-1)\sin\left(x+(n-2)\frac{\pi}{2}\right)$$

$$= x^2\sin\left(x+\frac{n\pi}{2}\right) - 2nx\sin\left(\pi/2 - \left(x+\frac{n\pi}{2}\right)\right) - n(n-1)\sin\left(\pi - \left(x+\frac{n\pi}{2}\right)\right)$$

$$= x^2\sin\left(x+\frac{n\pi}{2}\right) - 2nx\cos\left(x+\frac{n\pi}{2}\right) - n(n-1)\sin\left(x+\frac{n\pi}{2}\right)$$

$$= (x^2 - n^2 + n)\sin\left(x+\frac{n\pi}{2}\right) - 2nx\cos\left(x+\frac{n\pi}{2}\right)$$