8. If $y = \sin(m\sin^{-1}x)$, prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$$

Solution: Here $y = \sin(m\sin^{-1}x)$

So, $y_1 = \cos(m\sin^{-1}x)\cdot\dfrac{m}{\sqrt{1-x^2}}$

or, $(1-x^2)y_1^2 = m^2\cos^2(m\sin^{-1}x)$

or, $(1-x^2)y_1^2 = m^2(1-\sin^2(m\sin^{-1}x))$

or, $(1-x^2)y_1^2 = m^2(1-y^2)$

Differentiating w.r.t x, we have (w.r.t means with respect to)

$$(1-x^2)2y_1y_2 - 2xy_1^2 = -2m^2yy_1$$

or, $(1-x^2)y_2 - xy_1 + m^2y = 0 \quad \ldots(1) \quad [\because y_1 \neq 0]$

Differentiating n times w.r.t x and applying Liebnitz's theorem on (1), we have

$$((1-x^2)y_2)_n - (xy_1)_n + (m^2y)_n = 0$$

or, $(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n$

$$- xy_{n+1} - {}^nC_1\cdot 1\cdot y_n + m^2y_n = 0$$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - (n^2-n)y_n - xy_{n+1} - ny_n + m^2y_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$

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9. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, Then prove that

$$x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$$

Solution: $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$

or, $\cos^{-1}\left(\frac{y}{b}\right) = n\log\left(\frac{x}{n}\right)$

or, $y = b\cos[n(\log x - \log n)]$ . . . (1)

Differentiating with respect to x, we have

$$y_1 = -b\sin[n(\log x - \log n)]\cdot\frac{n}{x}$$

or, $xy_1 = -nb\sin[n(\log x - \log n)]$

or, $x^2y_1^2 = n^2b^2\sin^2[n(\log x - \log n)]$

$= n^2b^2(1 - \cos^2[n(\log x - \log n)])$

$= n^2b^2 - n^2y^2$ (from (i))

Differentiating w.r.t x, we have

$$x^2 2y_1y_2 + 2xy_1^2 = -2n^2yy_1$$

or, $x^2y_2 + xy_1 + n^2y = 0$ . . (2) [As $y_1 \neq 0$]

Differentiating (2) w.r.t x and applying Leibnitz's theorem, we get

$$(x^2y_2)_n + (xy_1)_n + (n^2y)_n = 0$$

or, $x^2y_{n+2} + {}^nC_1\cdot 2x\cdot y_{n+1} + {}^nC_2\cdot 2\cdot y_n + xy_{n+1} + {}^nC_1\cdot 1\cdot y_n + n^2y_n = 0$

or, $x^2y_{n+2} + 2nxy_{n+1} + (n^2-n)y_n + xy_{n+1} + ny_n + n^2y_n = 0$

or, $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$

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10. If $y = (\sin^{-1}x)^2$, prove that

$$(1-x^2)y_{n+2} - (2n+1)y_{n+1} - n^2y_n = 0$$

Solution : Here $y = (\sin^{-1}x)^2$

Then $y_1 = 2(\sin^{-1}x)\cdot\frac{1}{\sqrt{1-x^2}}$

or, $(1-x^2)y_1^2 = 4(\sin^{-1}x)^2$

or, $(1-x^2)y_1^2 = 4y$

Differentiating w.r.t x, we have

$$(1-x^2)2y_1y_2 - 2x.y_1^2 = 4y_1$$

or, $(1-x^2)y_2 - xy_1 - 2 = 0$ (As $y_1 \neq 0$) ... (1)

Differentiating (1) n times w.r.t x, we get and applying Leibnitz's theorem, we get

$$((1-x^2)y_2)_n - (xy_1)_n + (-2)_n = 0$$

or, $(1-x^2)y_{n+2} + {}^nC_1(-2x)y_{n+1} + {}^nC_2(-2)y_n - xy_{n+1} - {}^nC_1\cdot 1\cdot y_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

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11. If $y = e^{m\sin^{-1}x}$ then prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

and hence find $y_n(0)$ (nth derivative of y with respect to x at $x=0$)

Solution: Here $y = e^{m\sin^{-1}x}$ ... (1)

$$y_1 = e^{m\sin^{-1}x}\cdot\frac{m}{\sqrt{1-x^2}} \quad ...(2)$$

or, $(1-x^2)y_1^2 = m^2(e^{m\sin^{-1}x})^2$

or, $(1-x^2)y_1^2 = m^2y^2$

Differentiating w.r.t x we have,

$$(1-x^2)\,2y_1y_2 - 2xy_1^2 = m^2 \cdot 2yy_1$$

or, $(1-x^2)y_2 - xy_1 - m^2y = 0 \quad \cdots (3)$

Differentiating (3) w.r.t x n times and applying Leibnitz's theorem, we get

$$\left((1-x^2)y_2\right)_n - (xy_1)_n - (m^2y)_n = 0$$

or, $(1-x^2)y_{n+2} + {}^nc_1(-2x)y_{n+1} + {}^nc_2(-2)y_n - xy_{n+1} - {}^nc_1 \cdot 1 \cdot y_n - m^2y_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0 \quad \cdots (4)$

Putting $x=0$ in (4), we get

$$y_{n+2}(0) = (n^2+m^2)y_n(0)$$

Putting $x=0$, in equations (1), (2) and (3)

$$y(0) = 1, \quad y_1(0) = m \quad \text{and} \quad y_2(0) = m^2$$

Putting $n = 1, 2, 3, 4, \cdots$ in equation (4), we get

$$y_3(0) = (1^2+m^2)y_1(0) = m(1^2+m^2)$$
$$y_4(0) = (2^2+m^2)y_2(0) = m^2(2^2+m^2)$$
$$y_5(0) = (3^2+m^2)y_3(0) = m(1^2+m^2)(2^2+m^2)$$

In general, $y_n(0) = m^2(2^2+m^2)(4^2+m^2)\cdots((n-2)^2+m^2)$, n even
$= m(1^2+m^2)(3^2+m^2)\cdots((n-2)^2+m^2)$, n odd

Homework: HW 5. If $y = \sin(m\sin^{-1}x)$, find $y_n(0)$

HW.6 If $y = \cos(m\sin^{-1}x)$, show that

(i) $(1-x^2)y_2 - xy_1 + m^2y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Also find $y_n(0)$.