Functions of two and three variables:

Here we denote the set of all real numbers by $\mathbb{R}$.

The set $\mathbb{R}\times\mathbb{R} = \mathbb{R}^2 = \{(x,y) : x\in\mathbb{R}, y\in\mathbb{R}\}$ is the two dimensional space or the plane

The set $\mathbb{R}\times\mathbb{R}\times\mathbb{R} = \mathbb{R}^3 = \{(x,y,z) : x\in\mathbb{R}, y\in\mathbb{R} \text{ and } z\in\mathbb{R}\}$ is the three dimensional space.

A function $f : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}^2$ is said to to be a real valued function of two variables.

Examples: 1. $f(x,y) = x^2+y^2$. Here $S = \mathbb{R}^2$

2. $f(x,y) = \sqrt{xy}$ Here $S = \{(x,y) : x\geq 0, y\geq 0\} \subseteq \mathbb{R}^2$

A function $f : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}^3$ is said to be a real valued function of three variables.

Examples 1. $f(x,y,z) = x^2 + xyz + z^2$. Here $S = \mathbb{R}^3$

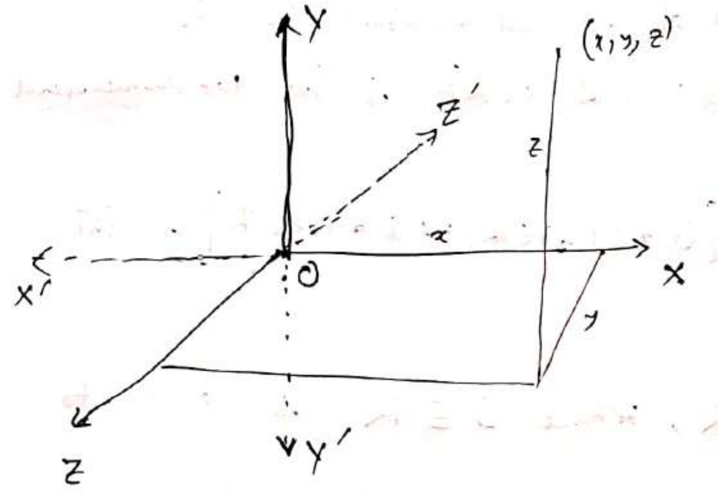
2. $f(x,y,z) = \sin^{-1}x + \sqrt{yz}$.

Here $S = \{(x,y,z) : -1\leq x\leq 1, y\geq 0, z\geq 0\} \subseteq \mathbb{R}^3$

Geometrical representation of function of two variables

Just as we represent functions of a single variable $f(x)$ by means of a curve $y = f(x)$ in a plane by representing (x,y) in a plane determined by a pair of rectangular axes, we represent a function of two variables $f(x,y)$ by means of a surface $z = f(x,y)$ in three dimensional space by representing (x,y,z) in the three dimensional space by taking three mutually perpendicular axes XOX', YOY' and ZOX' and representing (x,y,z)

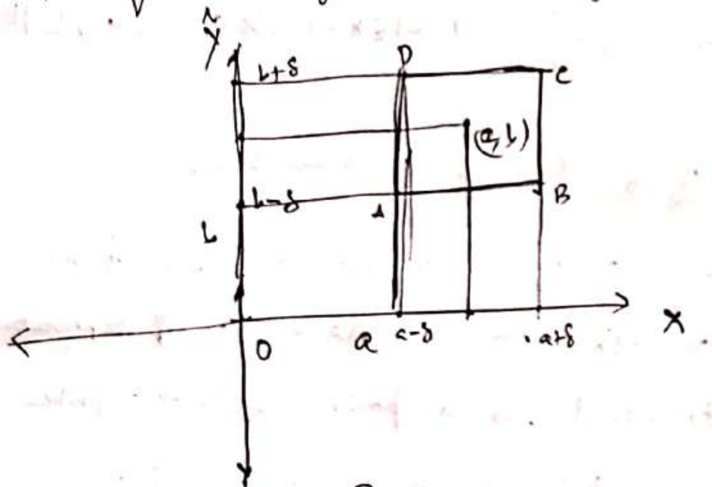
with respect to these axes



Limit and Continuity of function of two variables:

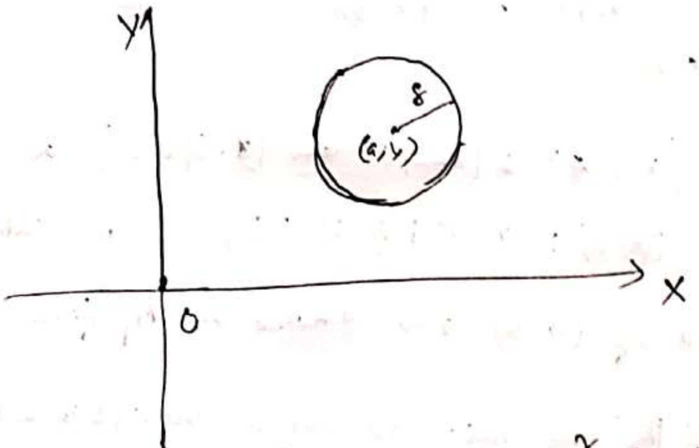
We begin with an important concept, the concept of neighbourhood of a point.

Definition 1 The set of (x, y) other than (a, b) that satisfy the inequalities $0<|x-a|<\delta$ and $0<|y-b|<\delta$ where δ is a small positive number, is said to form a square deleted δ-neighbourhood of the point (a, b).



i.e., the set $S=\{(x,y): a-\delta<x<a+\delta \text{ and } x\neq a,\ b-\delta<x<b+\delta \text{ and } x\neq b\}$ represented by the interior of the square ABCD excluding the point (a, b) is a square deleted δ-neighbourhood of (a, b)

Definition 2 : The set of (x, y) other than (a, b) that satisfy the condition $0 < (x-a)^2 + (y-b)^2 < \delta^2$ where δ is a small positive number is said to form a circular δ-neighbourhood of the point (a, b)



i.e., the set $S = \{ (x,y) : 0 < (x-a)^2 + (y-b)^2 < \delta^2 \}$ represented by the ~~cross~~ interior of the circle $(x-a)^2 + (y-b)^2 = \delta^2$ other than the centre is circular deleted δ-neighbourhood of the point (a, b)

Note : we denote a deleted δ-neighbourhood by $N((a,b), \delta)$ for Definition 1 & 2.

Definition : Limit : Let $f : S \to \mathbb{R}$, $S \subseteq \mathbb{R}^2$

Let (a, b) be a limit point of S (i.e., for every neighbourhood M of (a, b), $S \cap M \neq \phi$)

The function $f(x, y)$ is said to have a limit l as $(x, y) \to (a, b)$, denoted by $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$ if given a positive number ϵ there exists a positive number δ such that $| f(x,y) - l | < \epsilon$

whenever $(x, y) \in N((a,b), \delta) \cap S$

i.e for $(x, y) \in S$ where $0 < (x-a)^2 + (y-b)^2 < \delta^2$

or $0 < |x-a| < \delta$ and $0 < |y-b| < \delta$

Definition: Continuity: Let $f: S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^2$. Let $(a,b) \in S$. f is continuous at (a,b) if (a,b) is a limit point of S and

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

Note 1. The set ~~$N_1((a,b),\delta)$~~ $\{(x,y): |x-a|<\delta \text{ and } |y-b|<\delta\}$ or the set $\{(x,y): (x-a)^2+(y-b)^2<\delta\}$ is called a δ-neighbourhood of (a,b) and denoted by $N_1((a,b),\delta)$

Note 2 A set $M \subseteq \mathbb{R}^2$ is said to be a neighbourhood of (a,b) if there exists a $\delta>0$ such that

$$N_1((a,b),\delta) \subset M$$

Note 3 ϵ (epsilon) and δ (delta) are two greek letters.

Example 1. verify $\displaystyle\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$

Solution: Let $\epsilon>0$ be given.

Now $\left|\dfrac{x^2-y^2}{x^2+y^2}\right| < 1$ when $(x,y) \neq (0,0)$

So, $\left|xy\dfrac{x^2-y^2}{x^2+y^2} - 0\right| = \left|xy\dfrac{x^2-y^2}{x^2+y^2}\right|$

$$= |x||y|\frac{|x^2-y^2|}{x^2+y^2} < |x||y| < \epsilon$$

for $0<|x|<\delta$ and $0<|y|<\delta$ for $\delta = \sqrt{\epsilon}$

So, $\displaystyle\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$ by definition.

Example 2 Let $f(x,y) = xy\dfrac{x^2-y^2}{x^2+y^2}$, $(x,y) \neq (0,0)$

$= 0$, $(x,y) = (0,0)$

Show that f is continuous at $(0,0)$