<u>Solution:</u> we have show earlier in example 1, $\lim_{(x,y)\to(0,0)} \cancel{f(x,y)} \, xy\frac{x^2-y^2}{x^2+y^2} = 0$.

So, $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

Hence, f is continuous at $(0,0)$.

<u>Exercise</u> 1. Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

2. Show that $\lim_{(x,y)\to(0,0)} (|x|+|y|) = 0$

Note : $\lim_{(x,y)\to(a,b)} f(x,y)$ is also denoted by $\lim_{\substack{x\to a\\ y\to b}} f(x,y)$

Exercise 3. Are the following functions continuous at $(0,0)$?

(i) $f(x,y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

(ii) $f(x,y) = |x| + |y|$

<u>Partial derivatives</u> :

Let $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^2$ be a function of two variables. Let $(a,b) \in S$

If y be held constant value b then $f(x,y)$ depends on x alone and is a function of a single variable x. Hence, its derivative with respect to x at $x=a$ may exist. If it does, its value is called the partial derivative of $f(x,y)$ with respect to x at (a,b)

and is denoted by $f_x(a,b)$. Other notations are $D_x f(a,b)$, $D_1 f(a,b)$, $\left(\frac{\partial f}{\partial x}\right)_{x=a, y=b}$ and $\left(\frac{\partial f}{\partial x}\right)_{(a,b)}$

When the point in question is evident from the context, we simply write f_x or $\frac{\partial f}{\partial x}$.

So, we have then

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}, \text{ if it exists}$$

Similarly, if x is held constant at a, then $f(a,y)$ becomes a function of single variable y and its derivative with respect to y at $y=b$ may exist. If it does, its value is called the partial derivative of $f(x,y)$ with respect y at (a,b) and is denoted by $f_y(a,b)$. Other notations are $D_y f(a,b)$ $D_2 f(a,b)$, $\left(\frac{\partial f}{\partial y}\right)_{x=a, y=b}$ and $\left(\frac{\partial f}{\partial y}\right)_{(a,b)}$

When the point is question is evident from the context, we simply write $\frac{\partial f}{\partial y}$ or f_y

So, we have then

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a,b)}{k}; \text{ if it exists.}$$

If partial derivative of $f(x,y)$ at each point $(a,b) \in S$, then we say $f_x(x,y)$ exists and is defined by $f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$

Similarly $f_y(x,y)$ can be defined

They are also functions on S to $\mathbb{R}$.

Examples : 1. Obtain $f_x(2,1)$ and $f_y(2,1)$ for

$$f(x,y) = \frac{x+y-1}{x+y+1}$$

Solution: $f_x(2,1) = \cancel{\lim_{h\to 0} f(2+h) - f(2,1)} \; \lim_{h\to 0} \dfrac{f(2+h,1) - f(2,1)}{h}$

$$= \lim_{h\to 0} \frac{\frac{2+h}{4+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{1}{2(4+h)} = \frac{1}{8}$$

Similarly, $f_y(2,1) = \lim_{k\to 0} \dfrac{f(2,1+k) - f(2,1)}{k}$

$$= \lim_{k\to 0} \frac{\frac{k+2}{k+4} - \frac{1}{2}}{k} = \frac{1}{8}$$

2. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ where $z = x^2 + xy + y^2$

Solution: Here $z = f(x,y) = x^2 + xy + y^2$

So, $\dfrac{\partial z}{\partial x} = f_x(x,y) = \lim_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$

$$= \lim_{h\to 0} \frac{\{(x+h)^2 + (x+h)y + y^2\} - (x^2 + xy + y^2)}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2 + hy}{h} = \lim_{h\to 0} (2x + h + y) = 2x + y$$

Similarly, $f_y(x,y) = 2y + x$

Note: Evaluation of partial derivatives will involve no new difficulty. One can always consider the given function to be a function of a single variable, the other variable or variables if ~~any~~ any, being treated as constants. So, with the previous knowledge of derivatives of functions of a single variable, the result will come very easily. Thus in this example, since $z = x^2+xy+y^2$, considering y as constant, we obtain $\frac{\partial z}{\partial x}$

$= \frac{d}{dx}(x^2) + y\frac{d}{dx}(x) + \frac{d}{dx}(y^2) = 2x+y$. and similarly considering x as constant we obtain $\frac{\partial z}{\partial y} = 2y+x$

3. Find f_x, f_y if $f(x,y) = \tan^{-1}(y/x)$

Solution: Considering y as a constant,

$$f_x = \frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times y\left(-\frac{1}{x^2}\right) = -\frac{y}{x^2+y^2}$$

Considering x as a constant,

$$f_y = \frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{1}{x} \times 1 = \frac{x}{x^2+y^2}$$

Generalisation: The function $f(x,y,z)$ of three independent variables x, y and z has three partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$. These are derivatives of functions of a single variable when two others are held constants.

So, $f_x(x,y,z) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$

Similar definitions may be constructed by $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

Partial Derivatives of higher order

The partial derivatives $f_x(x,y)$ and $f_y(x,y)$ of the function $f(x,y)$ may ~~themselves~~ themselves be functions of x and y