and hence each may again admit of partial differentiation with respect to x and y. ~~and hence each may again~~ The partial derivative of $f_x(x,y)$ with respect to x will be given by

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \lim_{h\to 0} \frac{f_x(x+h,y) - f_x(x,y)}{h}$$

Similarly, partial derivative of $f_x(x,y)$ with respect to y will be given by $f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \lim_{k\to 0} \frac{f_x(x,y+k) - f_x(x,y)}{k}$

Similarly,

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \lim_{k\to 0} \frac{f_y(x,y+k) - f_y(x,y)}{k}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \lim_{h\to 0} \frac{f_y(x+h,y) - f_y(x,y)}{h}$$

The four partial derivatives f_{xx}, f_{yx}, f_{xy}, ~~are~~ f_{yy} are the second order partial derivatives. Third order and higher order partial derivatives are defined in a similar manner.

1. Examples: 1. If $u = e^{xyz}$, prove that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$$

Solution: $\frac{\partial u}{\partial z} = e^{xyz} \cdot xy$

So, $\frac{\partial^2 u}{\partial y \partial z} = (e^{xyz} \cdot xz)xy + xe^{xyz} = e^{xyz}(x^2yz + x)$.

So, $\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} \cdot yz(x^2yz + x) + e^{xyz}(2xyz + 1)$

$$= (1 + 3xyz + x^2y^2z^2)e^{xyz}.$$

Example 5 Verify that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{where } f(x,y) = \log\left(\frac{x^2+y^2}{xy}\right)$$

Solution: Here $f(x,y) = \log\left(\frac{x^2+y^2}{xy}\right) = \log(x^2+y^2) - \log(xy)$

$$= \log(x^2+y^2) - \log x - \log y$$

So, $$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x} = \frac{x^2-y^2}{x(x^2+y^2)}$$

and $$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} - \frac{1}{y} = \frac{y^2-x^2}{y(x^2+y^2)}$$

Hence, $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{1}{y}\left[\frac{(x^2+y^2)(-2x) - (y^2-x^2).2x}{(x^2+y^2)^2}\right]$$

$$= \frac{-4xy}{(x^2+y^2)^2}$$

Also $$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{1}{x}\left[\frac{(x^2+y^2)(-2y) - (x^2-y^2)2y}{(x^2+y^2)^2}\right]$$

$$= \frac{-4xy}{(x^2+y^2)^2}$$

So, $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Example 6 If $u = \log(x^3+y^3+z^3-3xyz)$, show that

(i) $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z}$$

(ii) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u = -\frac{3}{(x+y+z)^2}$$

(iii) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

Solution: (i) Since $x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

where ω and ω^2 are the imaginary cube roots of unity.

So, $u=\log(x+y+z)+\log(x+\omega y+\omega^2 z)+\log(x+\omega^2 y+\omega z)$

So, $\dfrac{\partial u}{\partial x}=\dfrac{1}{x+y+z}+\dfrac{1}{x+\omega y+\omega^2 z}+\dfrac{1}{x+\omega^2 y+\omega z}$

$\dfrac{\partial u}{\partial y}=\dfrac{1}{x+y+z}+\dfrac{\omega}{x+\omega y+\omega^2 z}+\dfrac{\omega^2}{x+\omega^2 y+\omega z}$

$\dfrac{\partial u}{\partial z}=\dfrac{1}{x+y+z}+\dfrac{\omega^2}{x+\omega y+\omega^2 z}+\dfrac{\omega}{x+\omega^2 y+\omega z}$

Hence $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial z}\right)u$

$=\dfrac{3}{x+y+z}+\dfrac{1+\omega+\omega^2}{x+\omega y+\omega^2 z}+\dfrac{1+\omega^2+\omega}{x+\omega^2 y+\omega z}$

$=\dfrac{3}{x+y+z}\quad\left[\because 1+\omega+\omega^2=0\right]$

(ii) $\cancel{\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}\right)u}$

$\dfrac{\partial^2 u}{\partial x^2}=-\dfrac{1}{(x+y+z)^2}-\dfrac{1}{(x+\omega y+\omega^2 z)^2}-\dfrac{1}{(x+\omega^2 y+\omega z)^2}$

$\dfrac{\partial^2 u}{\partial y^2}=-\dfrac{1}{(x+y+z)^2}-\dfrac{\omega}{(x+\omega y+\omega^2 z)^2}-\dfrac{\omega^2}{(x+\omega^2 y+\omega z)^2}$

$\dfrac{\partial^2 u}{\partial z^2}=-\dfrac{1}{(x+y+z)^2}-\dfrac{\omega^2}{(x+\omega y+\omega^2 z)^2}-\dfrac{\omega}{(x+\omega^2 y+\omega z)^2}$

So, $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}\right)u$

$=-\dfrac{3}{(x+y+z)^2}\quad\left(\because 1+\omega+\omega^2=1+\omega^2+\omega=0\right)$

(iii) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$

$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$

$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$ $\left[\because \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u=\frac{3}{x+y+z} \text{ from (i)}\right]$

$=3\left(\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right)$

$=3\left(-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}-\frac{1}{(x+y+z)^2}\right)$

$= -\frac{9}{(x+y+z)^2}$

Note: We have seen earlier that $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$ for the function in example 6. For most of the function, it is true. But there are functions where they are not equal. We give an example

Example 7 Show that

$\frac{\partial^2 f}{\partial x\partial y}\neq\frac{\partial^2 f}{\partial x\partial y}$ at $(0,0)$ for the function f

where $f(x,y) = xy\,\frac{x^2-y^2}{x^2+y^2}$, ~~(x,y) ≠ (0,0)~~ $x\neq 0, y\neq 0$

$= 0$, ~~(x,y) = (0,0)~~ otherwise.

Solution We first find out $\frac{\partial^2 f}{\partial x\partial y}$ at $(0,0) = f_{xy}(0,0)$

Now $f_{xy}(0,0)=\lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}$