Also, $f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k\to 0} \frac{hk\frac{h^2-k^2}{h^2+k^2} - 0}{k} = \lim_{k\to 0} h\frac{h^2-k^2}{h^2+k^2} = h$

and $f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$

So, $f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1$

We now find out $\frac{\partial^2 f}{\partial y\,\partial x}$ at $(0,0) = f_{yx}(0,0)$

Now $f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$

Now $f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h\to 0} \frac{hk\frac{h^2-k^2}{h^2+k^2} - 0}{h} = -k$

and $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$

So, $f_{yx}(0,0) = \lim_{k\to 0} \frac{-k-0}{k} = -1$

So, $f_{xy}(0,0) \neq f_{yx}(0,0)$

Note: When $f_{xy} = f_{yx}$, we say commutative property for the mixed partial derivatives hold.

Now we state Schwarz's theorem for commutative property of the mixed partial derivatives without proof.

~~Schwarz~~ Schwartz's theorem:

Let $f: S \to \mathbb{R}$ where S is an open set in $\mathbb{R}^2$. If $(a,b) \in S$ and (i) f_x, f_y exist in some neighbourhood of (a,b)

(ii) one of the second order ~~of~~ mixed derivative, say

f_{xy} is continuous at (a,b),

then $f_{yx}(a,b)$ exists and $f_{xy}(a,b) = f_{yx}(a,b)$

Euler's Theorem on homogeneous function of two and three variables

Definition: A real valued function $f(x,y)$ of two variables is said to be homogeneous of degree n if $f(tx,ty) = t^n f(x,y)$ for every positive t

An equivalent definition is, f is a homogeneous function of two variables of degree n if $f(x,y) = x^n \phi(y/x)$ or

$f(x,y) = y^n \psi(y/x)$

For example, $f(x,y) = x^2+y^2$ is a homogeneous function of degree 2 as $f(tx,ty) = (tx)^2+(ty)^2 = t^2(x^2+y^2)$ for any positive t or, $f(x,y) = x^2\left(1+y^2/x^2\right) = x^2 \phi\left(y/x\right)$ where $\phi(y/x) = 1+\left(\frac{y}{x}\right)^2$ or, $f(x,y) = y^2\left(1+\left(\frac{x}{y}\right)^2\right)$

$= y^2\psi\left(\frac{x}{y}\right)$ where $\psi\left(\frac{x}{y}\right) = 1+\left(\frac{x}{y}\right)^2$

Similarly $f(x,y) = x^2/y + \frac{y^2}{x}$ is a homogeneous function of degree 1 as $f(tx,ty) = \frac{(tx)^2}{ty} + \frac{(ty)^2}{tx}$

$$= t\left(\frac{x^2}{y}+\frac{y^2}{x}\right) = t\,f(x,y)$$

A real valued function $f(x,y,z)$ of three variables is said to a homogeneous function of degree n if $f(tx,ty,tz) = t^n f(x,y,z)$ for any positive t or equivalently if $f(x,y,z) = x^n \phi\left(\frac{y}{x}, \frac{z}{x}\right)$

For example, $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ is a homogeneous function of degree -1 as

$$f(tx,ty,tz) = \frac{1}{\sqrt{(tx)^2+(ty)^2+(tz)^2}} = \frac{1}{t}\,\frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$= t^{-1} f(x,y,z)$$

$$\text{or, } f(x,y,z) = \frac{1}{x}\cdot\frac{1}{\sqrt{1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2}}$$

$$= x^{-1}\phi\left(\frac{y}{x}, \frac{z}{x}\right)$$

where $\phi\left(\frac{y}{x}, \frac{z}{x}\right) = \frac{1}{\sqrt{1+\left(\frac{y}{x}\right)^2+\left(\frac{z}{x}\right)^2}}$

Theorem 1 (Euler's Theorem on homogeneous function of two variables): If $u = f(x,y)$ be a homogeneous function of x and y of degree n having continuous partial derivatives, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

Proof: Since $u = f(x,y)$ is a homogeneous function of degree n, we may express $u = x^n\phi(y/x)$

So, $\frac{\partial u}{\partial x} = nx^{n-1}\phi(y/x) + x^n\phi'\left(\frac{y}{x}\right)\times\left(\frac{-y}{x^2}\right)$

So, $x\frac{\partial u}{\partial x} = nx^n\phi\left(\frac{y}{x}\right) - x^{n-1}y\,\phi'\left(\frac{y}{x}\right)$

Again, $\frac{\partial u}{\partial y} = x^n\phi'\left(\frac{y}{x}\right)\times\frac{1}{x}$. So, $y\frac{\partial u}{\partial y} = x^{n-1}y\,\phi'\left(\frac{y}{x}\right)$

So, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nx^n\phi\left(\frac{y}{x}\right) = nu = n\,f(x,y)$

Similarly, we'll state the Euler's Theorem without proof for function of three variables.

<u>Theorem 2</u> (Euler's Theorem on homogeneous function of three variables): If $u = f(x,y,z)$ be a homogeneous function of degree n in three variables x, y, z, having continuous partial derivatives, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = nu$$

<u>Corollary of Theorem 1</u>: If u be a homogeneous function of x and y of degree n having second order continuous partial derivatives, then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

[Note: $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$ is also written as $\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 u$]

Proof: Since u is a homogeneous function of degree n having second order ~~of continuous~~ continuous ~~part~~ partial derivatives, we have by Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu \quad \dots (1)$$