Differentiating (1) with respect to x partially,

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x} \quad \cdots (2)$$

Differentiating (1) with respect to y partially,

$$x\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y} \quad \cdots (3)$$

Multiplying (2) by x and (3) by y and using the fact that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ (as second order partial derivatives are continuous) and adding, we get

$$\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right) + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = n\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

or, $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)nu = n(n-1)u$$

Some worked Examples

1. If $z = \tan^{-1}\frac{y}{x}$, verify that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Solution : Given $z = \tan^{-1}\frac{y}{x}$, so,

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

and $$\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2}\times\frac{1}{x} = \frac{x}{x^2+y^2}$$

So, $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\cdot 1 - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

and $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$

$$= -\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= -\frac{(x^2+y^2)\cdot 1 - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

So, $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$

2. If $z = f(x+ay) + \phi(x-ay)$, prove that

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Solution: Given $z = f(x+ay) + \phi(x-ay)$

So, $\dfrac{\partial z}{\partial x} = f'(x+ay) + \phi'(x-ay)$

and $\dfrac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay) \quad \ldots (1)$

$\dfrac{\partial z}{\partial y} = a f'(x+ay) - a\phi'(x-ay)$

and $\dfrac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 \phi''(x-ay)$

So, $\dfrac{\partial^2 z}{\partial y^2} = a^2 \left(f''(x+ay) + \phi''(x-ay) \right)$

or, $\dfrac{\partial^2 z}{\partial y^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$ (from (1))

3. Verify Euler's theorem for

(a) $z = ax^2 + 2hxy + by^2$ (b) $z = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

(c) $z = x^n \log \dfrac{y}{x}$

Solution: (a) Given $z = ax^2 + 2hxy + y^2$

$$= x^2 \left(a + 2h\frac{y}{x} + \left(\frac{y}{x}\right)^2 \right)$$

$$= x^n \phi\left(\frac{y}{x}\right) \text{ where } \phi\left(\frac{y}{x}\right) = a + 2h\frac{y}{x} + \left(\frac{y}{x}\right)^2$$

So, z is a homogeneous function of degree 2

and $\dfrac{\partial z}{\partial x} = 2ax + 2hy$

$\dfrac{\partial z}{\partial y} = 2hx + 2by$

So, $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2ax^2 + 2hxy + 2hxy + 2by^2$

$$= 2ax^2 + 4hxy + 2by^2$$

$$= 2(ax^2 + 2hxy + by^2)$$

$$= 2z$$

(b) Given $z = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

Let $f(x,y) = z = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$ Now $f(tx, ty) = \dfrac{(tx)^{1/4} + (ty)^{1/4}}{(tx)^{1/5} + (ty)^{1/5}}$

$$= t^{1/4 - 1/5} \cdot \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$

$= t^{1/20} f(x,y)$. So, z is a homogeneous function of degree $\frac{1}{20}$

Now $z = \dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$

So, $\log z = \log\left(x^{1/4}+y^{1/4}\right) - \ln\left(x^{1/5}+y^{1/5}\right)$

So, $\dfrac{1}{z}\dfrac{\partial z}{\partial x} = \dfrac{\frac{1}{4}x^{-3/4}}{x^{1/4}+y^{1/4}} - \dfrac{\frac{1}{5}x^{-4/5}}{x^{1/5}+y^{1/5}}$ ... (1)

and $\dfrac{1}{z}\dfrac{\partial z}{\partial y} \quad \dfrac{\frac{1}{4}y^{-3/4}}{x^{1/4}+y^{1/4}} - \dfrac{\frac{1}{5}y^{-4/5}}{x^{1/5}+y^{1/5}}$ ... (2)

~~Adding (1) and (2), we get~~

So, $\dfrac{1}{z}x\dfrac{\partial z}{\partial x} = \dfrac{\frac{1}{4}x^{1/4}}{x^{1/4}+y^{1/4}} - \dfrac{\frac{1}{5}x^{1/5}}{x^{1/5}+y^{1/5}}$ ... (3)

and $\dfrac{1}{z}y\dfrac{\partial z}{\partial y} = \dfrac{\frac{1}{4}y^{1/4}}{x^{1/4}+y^{1/4}} - \dfrac{\frac{1}{5}y^{1/5}}{x^{1/5}+y^{1/5}}$ ..(4)

Adding (3) and (4), we get

$$\frac{1}{z}\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right) = \frac{1}{4}-\frac{1}{5} = \frac{1}{20}$$

So, $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \dfrac{1}{20}z$

(c) Do it yourself.

4. If $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ $\quad x^2+y^2+z^2 \neq 0$,

Prove that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$