The method of determining the rank of a matrix by determining the value of the minors is much laborious and therefore we need some easier method for this purpose; with this aim, we introduce some transformations on the matrix, called the elementary operations on the matrix. When applied to rows, the elementary operations are said to be elementary row operations and when applied to columns, they are said to be elementary column operations. R_i will denote ith row. That means R_1 is first row, R_2 is second row etc. C_i will denote ith column. That means C_1 is the first column, C_2 is the second column etc.

An elementary operation on a matrix A is an operation of the following three types:

1. Interchange of any two rows or columns. They are denoted by $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$ (interchange of ith and jth row or interchange of ith and jth column) (they are also denoted by R_{ij} or C_{ij})
2. Multiplication of the ith row or ith column by a non-zero number k and denoted by kR_i or kC_i (they are also denoted by $R_i(k)$ or $C_i(k)$)
3. Addition of k times the jth row or column to the ith row or column. They are denoted by $R_i + kR_j$ or $C_i + kC_j$ (they are also denoted by $R_{ij}(k)$ or $C_{ij}(k)$)

It can be proved that the rank of a matrix remains unchanged by a finite number of above elementary operations.

Now, if an $m \times n$ matrix A can be reduced to

$\begin{bmatrix} I_r & O_{r\times(n-r)} \\ O_{(m-r)\times r} & O_{(m-r)\times(n-r)} \end{bmatrix}$ (called the normal form) by a finite number of elementary operations where I_r is the identity matrix of order and $O_{r\times(n-r)}$, $O_{(m-r)\times r}$ and $O_{(m-r)\times(n-r)}$ are all zero matrices of order $r\times(n-r)$, $(m-r)\times r$ and $(m-r)\times(n-r)$.

Consider the example $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & O_{3\times1} \\ O_{1\times3} & O_{1\times1} \end{bmatrix}$.

So, the matrix is in normal form where I_3 is the identity matrix of order 3, $O_{3\times1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $O_{1\times3} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and $O_{1\times1} = [0]$.

We also write the normal form as $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ where O's are the zero matrices of suitable order.

From this form, we can easily find the rank of the matrix. The rank of the matrix $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ is r because every minor of order greater than r must contain a zero row or a zero column, so it has value zero. and $\det(I_r) = 1$ ($\det(I_r)$ means the determinant of I_r). So, rank of $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ is r.

As elementary operation does not change the rank, So, rank of A is r.

Note: we write $A \sim B$ if A and B have the same rank.

Examples 4 Show that the rank of the matrix

$$A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix} \text{ is } 3$$

Solution: Here $A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ $\frac{1}{8}c_1, \frac{1}{2}c_4$.

$$\sim \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 3 & 2 & 1 \\ -1 & -1 & -3 & 2 \end{bmatrix} \quad \begin{matrix} c_2 - c_1 \\ c_3 - 3c_1 \\ c_4 - 3c_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 \\ -1 & 0 & 0 & 5 \end{bmatrix} \quad \frac{1}{3}c_2, \frac{1}{3}c_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 5 \end{bmatrix} \quad \begin{matrix} c_3 - c_2 \\ c_4 - c_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 5 \end{bmatrix} \quad R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \frac{1}{5}c_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad c_3 \leftrightarrow c_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & O \end{bmatrix}.$$

Where $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ So, rank of $A = 3$

5. Find the rank of the ~~following~~ matrix A where

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{bmatrix}$$ by using elementary operations.

Solution:

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{bmatrix} \begin{matrix} C_2 - 2C_1 \\ C_3 + C_1 \\ C_4 - 3C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -2 & 1 \\ -1 & 0 & -2 & 1 \end{bmatrix} R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -2 & 1 \\ -3 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 + 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} C_3 + 2C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} C_2 \leftrightarrow C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$$

So, rank of A = 2

Home work: HW 3. Find the rank of the following matrices by using elementary operations:

(i) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 7 & 8 \\ 3 & 6 & 9 & 12 & 15 \\ 4 & 8 & 12 & 14 & 16 \end{bmatrix}$

HW 4. Show that ~~rank~~ $r(A) = 2$ and $r(B) = 2$ ($r(A)$ means rank of A) where $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 & 3 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

but $r(AB) = 1$.