Note: Thus we see that, in general orthogonal transformation, we substitute for x and y expression in x' and y' of the first degree. Hence, by this substitution, the degree of the equation is not altered, since the transformation is linear.

1.6 Invariants in orthogonal transformation

Relations connecting the coefficient of an expression or some other quantity which remains unchanged under an orthogonal transformation are called invariants under the orthogonal transformation.

If, by the orthogonal transformation without change of origin, the expression $ax^2+2hxy+by^2+2gx+2fy+c$ be changed into $a'x'^2+2h'x'y'+b'y'^2+2g'x'+2f'y'+c'$, then

(i) $a'+b'=a+b$ (ii) $a'b'-h'^2=ab-h^2$

(iii) $g'^2+f'^2=g^2+f^2$ (iv) $a'b'+b'c'+c'a'-f'^2-g'^2-h'^2$
$= ab+bc+ca-f^2-g^2-h^2$

Proof: By applying the formulae for orthogonal transformation,

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta,$$

the given expression changes to

$$a(x'\cos\theta-y'\sin\theta)^2+2h(x'\cos\theta-y'\sin\theta)(x'\sin\theta+y'\cos\theta)$$
$$+\,b(x'\sin\theta+y'\cos\theta)^2+2g(x'\cos\theta-y'\sin\theta)$$
$$+\,2f(x'\sin\theta+y'\cos\theta)+c$$

$$=(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)x'^2+2\{h(\cos^2\theta-\sin^2\theta)-(a-b)\sin\theta\cos\theta\}x'y'$$
$$+\,(a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta)y'^2$$
$$+\,2(g\cos\theta+f\sin\theta)x'+2(f\cos\theta-g\sin\theta)y'+c$$

so that

$$a' = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$$

$$h' = h(\cos^2\theta - \sin^2\theta) - (a-b)\sin\theta\cos\theta$$

$$b' = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta$$

$$g' = g\cos\theta + f\sin\theta, \quad f' = f\cos\theta - g\sin\theta, \quad c' = c$$

Hence (i) $a' + b' = a + b$ since $\sin^2\theta + \cos^2\theta = 1$

(ii) $2a' = a(1+\cos 2\theta) + 2h\sin 2\theta + b(1-\cos 2\theta)$

$$= (a+b) + \{2h\sin 2\theta + (a-b)\cos 2\theta\}$$

Similarly $2b' = (a+b) - \{2h\sin 2\theta + (a-b)\cos 2\theta\}$

So, $4a'b' = 2a' \cdot 2b' = (a+b)^2 - \{2h\sin 2\theta + (a-b)\cos 2\theta\}^2$

Also $4h'^2 = \{2h\cos 2\theta - (a-b)\sin 2\theta\}^2$

So $4(a'b' - h'^2) = (a+b)^2 - \{4h^2 + (a-b)^2\}$

$$= 4(ab - h^2)$$

So, $a'b' - h'^2 = ab - h^2$

(iii) $g'^2 + f'^2 = (g\cos\theta + f\sin\theta)^2 + (f\cos\theta - g\sin\theta)^2$

$$= g^2(\cos^2\theta + \sin^2\theta) + f^2(\sin^2\theta + \cos^2\theta)$$

$$= g^2 + f^2$$

(iv) $a'b' + b'c' + c'a' - f'^2 - g'^2 - h'^2$

$$= c'(a'+b') + (a'b' - h'^2) - (f'^2 + g'^2)$$

$$= c(a+b) + (ab - h^2) - (f^2 + g^2) \quad \text{from (i), (ii) \& (iii)}$$

$$= ab + bc + ca - f^2 - g^2 - h^2$$

Note: $a' + b' + c' = a + b + c$ also.

Worked Examples

1. Find the transformed equation of the straight line $\frac{x}{a}+\frac{y}{b}=2$ when the origin is transferred to the point (a,b)

Solution : The transformation formulae are

$x=x'+a,\ y=y'+b$, so that the transformed equation is

$$\frac{1}{a}(x'+a)+\frac{1}{b}(y'+b)=2 \quad \text{or, } \frac{x'}{a}+\frac{y'}{b}=0$$

2. Find the transformed equation when the axes are transformed to the parallel axes through the point $(2,-3)$, to the equation $2x^2+4xy+3y^2-2x-4y+7=0$

Solution : The transformation formulae are

$$x=x'+2,\ y=y'-3$$

So, the transformed equation is

$$2(x'+2)^2+4(x'+2)(y'-3)+3(y'-3)^2-2(x'+2)-4(y'-3)+7=0$$

$$\text{or, } 2x'^2+4x'y'+3y'^2-6x'-14y'+26=0$$

3. Find the transformed equation of the equation $x^2+2\sqrt{3}\,xy-y^2-2=0$ when the axes are rotated through an angle $30°$.

Solution: The transformation formulae are

$$x=x'\cos 30°-y'\sin 30°=\frac{1}{2}(x'\sqrt{3}-y')$$

$$y=x'\sin 30°+y'\cos 30°=\frac{1}{2}(x'+y'\sqrt{3})$$

The transformed equation is

$$(x'\sqrt{3}-y')^2+2\sqrt{3}(x'\sqrt{3}-y')(x'+y'\sqrt{3})-(x'+y'\sqrt{3})^2-8=0$$

or, $8x'^2 - 8y'^2 = 8$ or $x'^2 - y'^2 = 1$

4. To what point the origin is to be moved so that one can get rid of the first degree terms from the equation

$$x^2 + xy + 2y^2 - 7x - 5y + 12 = 0 \; ?$$

Solution: Let the point to which the origin is to be shifted be (α, β). Substituting $x = x' + \alpha$, $y = y' + \beta$, the equation becomes

$$(x'+\alpha)^2 + (x'+\alpha)(y'+\beta) + 2(y'+\beta)^2 - 7(x'+\alpha) - 5(y'+\beta) + 12 = 0$$

The coefficients of x' and y' in the transformed equation are $(2\alpha + \beta - 7)$ and $(\alpha + 4\beta - 5)$,
which will be seperately zero, if the first degree terms are to be removed.

Thus $2\alpha + \beta - 7 = 0$ and $\alpha + 4\beta - 5 = 0$

Solving, we get $\alpha = \frac{23}{7}$, $\beta = \frac{3}{7}$

Hence the origin must be shifted to the point $\left(\frac{23}{7}, \frac{3}{7}\right)$

Note: To remove the linear terms, the origin is to be shifted

5. Find the angle through the axes are to be ~~rotated~~ rotated so that the equation

$x\sqrt{3} + y + 6 = 0$ may be reduced to the form $x = c$. Also determine the value of c.

Solution: Let the axes be rotated through an angle θ,