so that $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$

The reduced equation becomes, after removing the primes

$$\sqrt{3}(x\cos\theta - y\sin\theta) + x\sin\theta + y\cos\theta + 6 = 0$$

$$\text{or, } x(\sqrt{3}\cos\theta + \sin\theta) + y(\cos\theta - \sqrt{3}\sin\theta) + 6 = 0$$

Since This is to be of the form $x = c$

So, $\cos\theta - \sqrt{3}\sin\theta = 0$, or $\tan\theta = \frac{1}{\sqrt{3}}$ or, $\theta = \frac{\pi}{6}$

So, the transformed equation is

$$x\left(\frac{3}{2} + \frac{1}{2}\right) + 6 = 0$$

or, $2x = -6$ or $x = -3$

Hence $c = -3$.

6. Find the angle by which the axes should be rotated so that the equation $ax^2 + 2hxy + by^2 = 0$ becomes another equation in which term xy is absent.

In particular, find the angle through which the axes are to be rotated so that the equation

$$17x^2 + 18xy - 7y^2 = 1$$

may be reduced to the form $Ax^2 + By^2 = 1$, $A > 0$; find also A and B.

Solution: Let the axes be turned through an angle θ. Then after removing the primes, the transformed equation becomes

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + b(x\sin\theta + y\cos\theta)^2 = 0$$

The coefficient of xy will be zero if

$$(b-a)\sin\theta\cos\theta + h(\cos^2\theta - \sin^2\theta) = 0$$

that is, $-\frac{1}{2}(a-b)\sin 2\theta + h\cos 2\theta = 0$

or, $\tan 2\theta = \frac{2h}{a-b}$, giving $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$

In the given case, $a = 17$ $h = 9$, $b = -7$

Therefore $\theta = \frac{1}{2}\tan^{-1}\frac{18}{17+7} = \frac{1}{2}\tan^{-1}\frac{3}{4}$

By invariants, we get

$$A+B = 17-7 = 10 \text{ and}$$

$$AB = 17(-7) - 9^2 = -200$$

Therefore $(A-B)^2 = (A+B)^2 - 4AB = 900$

or, $A-B = \pm 30$

Hence $A = 20$, $B = \cancel{00}\ -10$ (Since $A>0$)

and the transformed equation is $20x^2 - 10y^2 = 1$

Note: To remove the xy-term, the axes are to be rotated.

More generally, to remove any second degree term, e.g., x^2, y^2, xy, the axes are to be rotated.

7. Determine the angle through which the axes must be turned so that the equation $lx+my+n=0$ $(l\neq 0)$ may reduce to the form $ax+b=0$.

Solution: Let $x = px' - qy'$ and $y = qx' + py'$ be the required rotation where $p = \cos\theta$, $q = \sin\theta$

Then $lx+my+n = l(px'-qy') + m(qx'+py') + n$

$= (lp+mq)x' + (mp-lq)y' + n$

If this is to take the form $ax+b$, we must have

$$mp - lq = 0 \quad \text{or,} \quad \frac{q}{p} = \frac{m}{l}$$

or $\tan\theta = \frac{m}{l}$. Hence $\theta = \tan^{-1}\frac{m}{l}$, which is the required angle.

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2. General equation of second degree in x and y

The general equation of second degree in x and y is usually written in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots (1)$$

The curve represented by (1) is called second order curve

Let us introduce the notation $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

and $D = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2$

Δ is called the discriminant of (1) and is invariant under translation and rotation of axes.

D is also invariant under translation and rotation of axes.

We can show that

if $\Delta = 0$, then the equation (1) represents a pair of straight lines.

These two lines are intersecting if $D \neq 0$ and parallel (or coincident) if $D = 0$

If $\Delta = 0$, then the equation (1) is said to represent a degenerate conic.

If $a = b$ and $h = 0$, then the equation (1) represents a circle.

If $\Delta \neq 0$, then the equation (1) represents either an ellipse or a hyperbola, or a parabola (Non-degenerate Conic). These are proper conics.

If $\Delta \neq 0$ and $D > 0$, equation (1) represents an ellipse (or circle)

If $\Delta \neq 0$ and $D < 0$, equation (1) represents a hyperbola

If $\Delta \neq 0$ and $D = 0$, equation (1) represents a parabola

Central Conic (Definition): If any chord of a conic through a particular point is bisected by the point then the conic is said to be central conic and that particular point is called the centre of this conic.

It can be shown that if the conic (1) is central and (α, β) be its centre,

$$\text{then} \quad a\alpha + h\beta + g = 0 \quad \ldots (2)$$
$$\text{and} \quad h\alpha + b\beta + f = 0 \quad \ldots (3)$$

From (2) and (3), we have

$$\alpha = \frac{fh - bg}{ab - h^2}, \quad \beta = \frac{gh - af}{ab - h^2}$$

So, a conic is a central conic if $ab - h^2 \neq 0$

So, ellipse or hyperbola is a central conic and parabola is a non-central conic.

Canonical form: The equation (1), that is, a general equation of second degree in x and y can be reduced to the standard equation of a conic by suitable transformation