of coordinates. The standard equation is also called the canonical or normal form of the conic.

To find the canonical form from the equation (1), the following transformations are made ~~successively~~ successively:

(i) The term in $xy$ is removed by suitable rotation of axes

(ii) One or both (when possible) the terms in $x$ and $y$ are removed by translation

(iii) The constant is removed, if possible.

This is called reduction to Canonical form or normal form.

[Note: When $\Delta = 0$ and $D \neq 0$, the equation (1) represents a pair of straight lines (imaginary or intersecting) or a point ellipse.
When $\Delta \neq 0$ and $D = 0$, the equation (1) represents a pair of parallel or coincident straight lines.]

Case 1 Equation (1) is a central conic (Ellipse or hyperbola)

Here $\Delta \neq 0$ and $D \neq 0$

By transferring the origin to the centre $(\alpha, \beta)$ where $a\alpha + h\beta + g = 0$ and $h\alpha + b\beta + f = 0$, the equation (1) reduces to

$$ax^2 + 2hxy + by^2 + d = 0 \quad \cdots (4)$$ (terms in $x$ and $y$ are removed)

where $d = g\alpha + f\beta + c = \dfrac{\Delta}{D}$

Now rotating the axes through an angle $\theta$ such that

$\tan 2\theta = \dfrac{2h}{a-b}$ or $\theta = \dfrac{1}{2}\tan^{-1}\dfrac{2h}{a-b}$, the equation (4) reduces to

$$Ax^2 + By^2 + d = 0 \quad \cdots (5)$$ (terms in $xy$ is removed)

From (5), we get the canonical form.

Case 2 Equation (1) is a non-central conic (Parabola).

Here $\Delta \neq 0$, $D = 0$

As $D = ab - h^2 = 0$, the second degree terms in Equation (1) will form a perfect square.

So, the equation (1), i.e.,

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can be written as

$(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$ or, $(\alpha x + \beta y)^2 = -2gx - 2fy - c$ .... (2)

Introducing a real number $\lambda$, in (2), we write (2) as

$$(\alpha x + \beta y)^2 + 2(\alpha x + \beta y)\lambda + \lambda^2 = 2(\alpha\lambda - g)x + 2(\beta\lambda - f)y + \lambda^2 - c$$

or, $(\alpha x + \beta y + \lambda)^2 = 2(\alpha\lambda - g)x + 2(\beta\lambda - f)y + \lambda^2 - c$ ... (4)

We choose $\lambda$ such that the two straight lines

$$\alpha x + \beta y + \lambda = 0 \quad \cdots (5)$$

and $2(\alpha\lambda - g)x + 2(\beta\lambda - f)y + \lambda^2 - c = 0$ ... (6)

are perpendicular.

So, $\alpha(\alpha\lambda - g) + \beta(\beta\lambda - f) = 0$

or, $$\lambda = \frac{\alpha g + \beta f}{\alpha^2 + \beta^2}$$

For this value of $\lambda$, (5) and (6) are perpendicular

The straight lines (5) and (6) are respectively the axis and the tangent at the vertex of the parabola for this value of $\lambda$. Solving (5) and (6) we get the vertex of the parabola.

Let us now choose these perpendicular straight lines as coordinate axes with reference to which a point $P(x,y)$ on the curve is $(X,Y)$.

Then $X$ = perpendicular distance of $P(x,y)$ from the straight line (6)

$$= \frac{2(\alpha\lambda-g)x+2(\beta\lambda-f)y+\lambda^2-c}{2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}} \quad \ldots \quad (7)$$

and $Y$ = the perpendicular distance of $P(x,y)$ from the straight line (5).

$$= \frac{\alpha x+\beta y+\lambda}{\sqrt{\alpha^2+\beta^2}} \quad \ldots \quad (8)$$

Thus, with reference to the new axes, equation (4) can be written as

$$(\alpha^2+\beta^2)Y^2 = 2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}\,X$$

~~which is the~~ or, $Y^2 = \dfrac{2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}}{\alpha^2+\beta^2}\cdot X$

which is the canonical form of a parabola whose axis is $Y=0$, i.e., $\alpha x+\beta y+\lambda=0$ and the tangent at the vertex is the $Y$ axis, i.e.,

$X=0$, or,

$$2(\alpha\lambda-g)+2(\beta\lambda-f)y+\lambda^2-c=0$$

The length of the latus rectum is

$$\frac{2\sqrt{(\alpha\lambda-g)^2+(\beta\lambda-f)^2}}{\alpha^2+\beta^2} = \frac{2(\alpha f-\beta g)}{(\alpha^2+\beta^2)^{3/2}}$$ (by putting the value of $\lambda$)

Worked Examples,

1. Reduce the following equation to canonical form and hence determine the nature of the conic:

$$3x^2 + 2xy + 3y^2 - 16x + 20 = 0$$

Solution: Comparing the equation $3x^2+2xy+3y^2-16x+20=0$ ... (1)

with $ax^2+2hxy+by^2+2gx+2fy+c=0$, we have

$a=3,\ b=3,\ h=1,\ g=-8,\ f=0$ and $c=20$

So, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc+2fgh-af^2-bg^2-ch^2$

$= 3.3.20 + 2.0.(-8).1 - 3.0^2 - 3(-8)^2 - 20.1^2$

$= -32 \neq 0$

and $D = ab-h^2 = 3.3-1^2 = 8 \neq 0$

So, equation (1) is a central conic. Let the centre be $(\alpha, \beta)$. Then $3\alpha+\beta-8=0$

and $\alpha + 3\beta = 0$

Solving, we have $\alpha = 3,\ \beta = -1$

So, translating the origin to the centre $(3,-1)$, the equation (1) becomes with $(x', y')$ as the new coordinates

$3x'^2 + 2x'y' + 3y'^2 + d = 0$ ... (2) (As terms in x and in y vanish)

Where $d = (-8)\times 3 + 0.(-1) + 20 = -4$

So, (2) can be written as

$$3x'^2 + 2x'y' + 3y'^2 - 4 = 0$$

To remove the $x'y'$ term, we rotate the axes through an angle $\theta$. Due to this rotation, the equation reduces to (with new coordinates $(X, Y)$)

$$3(X\cos\theta - Y\sin\theta)^2 + 2(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + 3(X\sin\theta + Y\cos\theta)^2 - 4 = 0$$