or, $(3+\sin 2\theta)X^2 + 2\cos 2\theta\, XY + \cancel{Y^2}(3-\sin 2\theta)\cancel{2}Y^2 = 4 \quad \cdots (3)$

For removal of XY term, we require that

$$\cos 2\theta = 0, \text{ that is } \theta = \frac{\pi}{4}$$

Then (3) becomes

$$\cancel{or}\ (3+1)X^2 + (3-1)Y^2 = 4$$

or, $$\frac{X^2}{1} + \frac{Y^2}{2} = 1$$

This is the reduced canonical form of the given equation.

This represents an ellipse.

2. Reduce the following equation to canonical form and hence determine the nature of the conic:

~~Solution Comparing the~~ $x^2+4xy+y^2-2x+2y+6=0$

Solution: Comparing the equation $x^2+4xy+y^2-2x+2y+6=0$ with the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$, we have $a=b=1$, $h=2$, $g=-1$, $f=1$, and $c=6$

So, $$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -1 & 1 & 6 \end{vmatrix} = 1\cdot1\cdot6 + 2\cdot1\cdot(-1)\cdot2 - 1\cdot1^2 - 1\cdot(-1)^2 - 6\cdot2^2 = -24 \neq 0$$

and $D = ab - h^2 = 1-4 = -3 \neq 0$

Hence it is a central conic. Let the centre be (α, β). Then

$$\alpha + 2\beta - 1 = 0$$

and $$2\alpha + \beta + 1 = 0$$

This gives $\alpha = -1$, $\beta = 1$

Also $d = g\alpha + f\beta + c$

$= (-1)(-1) + 1\cdot 1 + 6 = 8$

The reduced equation with the centre at the origin is

or, $x'^2 + 4x'y' + y'^2 + d = 0$

or, $x'^2 + 4x'y' + y'^2 + 8 = 0$.

If the finally reduced equation after rotation be

$AX^2 + BY^2 + 8 = 0$

then by the theory of invariants, we have

$A + B = 1 + 1$ and $AB = 1 - 4 = -3$

So, $A = -1, 3$ and corresponding $B = 3, -1$

Hence the reduced equation to the canonical form of the given equation is

either $-X^2 + 3Y^2 + 8 = 0$, i.e., $\dfrac{X^2}{8} - \dfrac{Y^2}{8/3} = 1$

or, $3X^2 - Y^2 + 8 = 0$, i.e., $\dfrac{Y^2}{8} - \dfrac{X^2}{8/3} = 1$,

each of which is a parabola.

Note: The transformation can also be effected by the rotation of axes through an angle θ, given by $\tan 2\theta = \dfrac{2h}{a-b}$ giving $\theta = \dfrac{\pi}{4}$ in this case

3. Reduce the following equation to

canonical form: $6x^2-5xy-6y^2+14x+5y+4=0$

Solution: Here $a=6$, $b=-6$, $h=-\frac{5}{2}$, $g=7$, $f=\frac{5}{2}$, and $c=4$

Here $\Delta=0$, and $D=-36-\frac{25}{4}=-\frac{169}{4}\neq 0$

Thus the given equation represents a pair of intersecting ~~straightlines~~ straight lines,

The equations giving their point of intersection (α,β),

is
$$a\alpha+h\beta+g=0$$
$$h\alpha+b\beta+f=0$$

or, $12\alpha-5\beta+14=0$

and $-5\alpha-12\beta+5=0$

So, $\alpha=-\frac{11}{13}$, $\beta=\frac{10}{13}$

Shifting the origin to the point (α,β) without changing the direction of the axes, the given equation reduces to

$$6x'^2-5x'y'-6y'^2+d=0$$

where $d=7\times\left(-\frac{11}{13}\right)+\frac{5}{2}\times\frac{10}{13}+4=0$

or, $6x'^2-5x'y'-6y'^2=0$ ... (1)

Rotating the axes through an angle θ, (1) reduces to

$$6(X\cos\theta-Y\sin\theta)^2-5(X\cos\theta-Y\sin\theta)(X\sin\theta+Y\cos\theta)-6(X\sin\theta+Y\cos\theta)^2=0$$

or, $\left(6\cos2\theta-\frac{5}{2}\sin2\theta\right)X^2-(5\cos2\theta+12\sin2\theta)XY$
$$-\left(6\cos2\theta-\frac{5}{2}\sin2\theta\right)Y^2=0$$

To remove XY term, we put $5\cos2\theta+12\sin2\theta=0$

or, $\frac{\sin2\theta}{-5}=\frac{\cos2\theta}{12}=\frac{1}{13}$

with this values, the given equation reduces to

$X^2 - Y^2 = 0$ which is represent the

Canonical form and represent two intersecting straight lines $X+Y=0$ and $X-Y=0$

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4. Reduce the following equation to canonical form:

$4x^2 - 4xy + y^2 + 2x - 26y + 9 = 0$

Solution Here $a = 4$, $b = 1$, $h = -2$, $g = 1$, $f = -13$, $c = 9$

So, $\Delta = -625 \neq 0$

$\Delta = 0$

Hence the given equation represents a parabola

The given equation may be written as

$(2x-y)^2 + 2x - 26y + 9 = 0$

or, $(2x - y + \lambda)^2 = -2x + 26y - 9 + \lambda^2 + 4\lambda x - 2\lambda y$

$= 2(2\lambda - 1)x + 2(13-\lambda)y + \lambda^2 - 9$, λ

being a constant.

λ is so chosen that the two straight lines

$2x - y + \lambda = 0$ and $2(2\lambda-1)x + 2(13-\lambda)y + \lambda^2 - 9 = 0$

are perpendicular.

This gives $2\left\{\frac{-(2\lambda-1)}{13-\lambda}\right\} = -1$

or, $\lambda = 3$

Then the given equation becomes

~~$2x-y+3$~~ $(2x-y+3)^2 = 10x + 20y = 10(x+2y)$