or, $\dfrac{(2x-y+3)^2}{2^2+1} = \dfrac{10}{\sqrt{5}} \dfrac{x+2y}{\sqrt{1+2^2}}$

Taking the two perpendicular straight lines

$$x+2y=0 \quad \text{and} \quad 2x-y+3=0 \quad \text{as the}$$

axes of coordinates, the formulae for transformation are $X = \dfrac{x+2y}{\sqrt{1+2^2}}$ and $Y = \dfrac{2x-y+3}{\sqrt{2^2+1}}$

and this equation reduces to the canonical form

$$Y^2 = \frac{10}{\sqrt{5}} X$$

3. Pair of straight lines

3.1 Equation of a pair of straight lines.

Consider the equations of two straight lines

$$a_1x + b_1y + c_1 = 0 \qquad \cdots \quad (1)$$

and $$a_2x + b_2y + c_2 = 0 \qquad \cdots \quad (2)$$

Let us consider the equation

$$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2) = 0 \qquad \cdots \quad (3)$$

The coordinates of any point on (1) satisfy the equation (3) and hence all points of the straight line (1) lie on (3). Similarly, all points of the straight line (2) lie also on (3). Hence equation (3) represents both the straight lines. On expanding (3) we get a second degree equation in x and y. We say that the equation (3) represents a pair of straight lines (1) and (2)

3.2 A Theorem

A homogeneous second degree equation in x and y always represents a pair of straight lines passing through origin.

Proof: Let us consider the homogeneous equation in x and y

$$ax^2 + 2hxy + by^2 = 0 \quad \ldots (1)$$

Now $ax^2 + 2hxy + by^2 = b\left(y^2 + \frac{2h}{b}xy + \frac{a}{b}x^2\right)$

$$= b(y - m_1x)(y - m_2x), \text{ where}$$

$$m_1 + m_2 = -\frac{2h}{b} \quad \text{and} \quad m_1m_2 = \frac{a}{b}$$

Thus the homogeneous second degree equation in x and y represents two straight lines $y - m_1x = 0$ and $y - m_2x = 0$ passing through the origin. The straight lines are real or imaginary according as the values of m_1 and m_2 are real or imaginary.

If, again $m_1 = m_2$, then ~~the~~ the two straight lines are coincident.

Writing the equation (1) as a quadratic equation in y, we have

$$by^2 + 2hxy + ax^2 = 0$$

Thus the two straight lines are given by

$$y = \frac{-h \pm \sqrt{h^2 - ab}}{b}x$$

Hence the straight lines are real, if $h^2 - ab \geqslant 0$

and imaginary, if $h^2 - ab < 0$

The equality refers to the case when the lines are coincident.

3.3 Angle between the pair of straight lines passing through the origin.

Suppose $y = m_1 x$ and $y = m_2 x$ are the individual straight lines as given by the pair represented by

$$ax^2 + 2hxy + by^2 = 0$$

Then $m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$

If θ be the angle between these two straight lines,

$$\tan^2\theta = \frac{(m_1 - m_2)^2}{(1+m_1m_2)^2} = \frac{(m_1+m_2)^2 - 4m_1m_2}{(1+m_1m_2)^2}$$

$$= \frac{\left(-\frac{2h}{b}\right)^2 - 4\cdot\frac{a}{b}}{\left(1+\frac{a}{b}\right)^2}$$

$$= \frac{4(h^2 - ab)}{(a+b)^2}$$

So, $\tan\theta = \pm \frac{2\sqrt{h^2 - ab}}{a+b}$ or, $\theta = \tan^{-1}\left(\pm \frac{2\sqrt{h^2-ab}}{a+b}\right)$

+ or – sign is taken according as θ is the acute or obtuse angle between the lines

Note 1 Condition of coincidence : If $\theta = 0$, the lines are coincident and in this case $h^2 = ab$.

Note 2 Condition for perpendicularity: If $\theta = \pi/2$, the lines are at right angle and in this case

$$\cot\theta = 0 = \frac{a+b}{2\sqrt{h^2-ab}} \quad \text{or,} \quad a+b=0.$$

3.4 Bisectors of the angles between the pair of straight lines $ax^2+2hxy+by^2=0$

If θ_1 and θ_2 be the angles that the straight lines A~~OB~~ make with positive direction of the x-axis, then the two straight lines AOA' and BOB' given by the above equation are respectively

$$y = \tan\theta_1 \cdot x \quad \text{and} \quad y = \tan\theta_2 \cdot x$$

So, $\tan\theta_1 + \tan\theta_2 = -\frac{2h}{b}$ and $\tan\theta_1 \cdot \tan\theta_2 = \frac{a}{b}$.

If θ be the angle that either bisector POP' or QOQ'

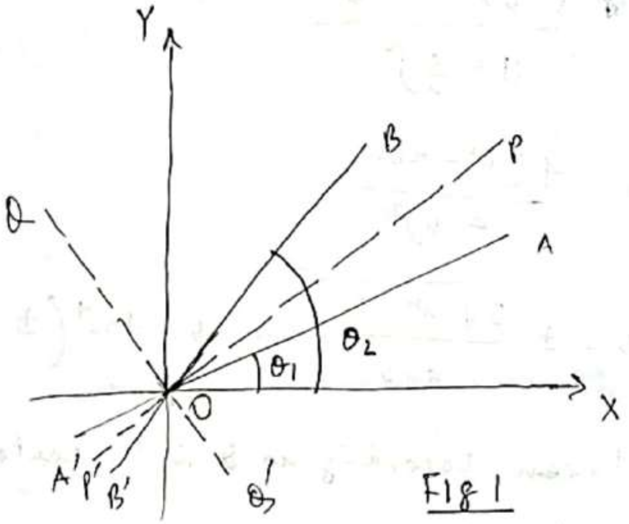

Fig 1

makes with the x-axis, then $\theta = \frac{1}{2}(\theta_1+\theta_2)$ or $\frac{1}{2}\pi + \frac{1}{2}(\theta_1+\theta_2)$. In either case, $\tan 2\theta = \tan(\theta_1+\theta_2)$