So, $\dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2} = \dfrac{-2h/b}{1-a/b} = \dfrac{2h}{a-b}$

If (x,y) be the coordinates of any point on either bisector, then $\tan\theta = \frac{y}{x}$

So, we get $\dfrac{2y/x}{1-y^2/x^2} = \dfrac{2h}{a-b}$

or, $\dfrac{x^2-y^2}{a-b} = \dfrac{xy}{h}$

This is the required equation of two bisectors.

Note 1 Taking θ acute in 3.3, we write $\tan\theta = \dfrac{2\sqrt{h^2-ab}}{a+b}$

3.5 Condition that the general equation of the second degree in x and y may represent a pair of straight line:

A necessary and sufficient condition that

$ax^2+2hxy+by^2+2gx+2fy+c=0$ may represent a pair of straight line is $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc+2fgh-af^2-bg^2-ch^2 = 0$

Proof: The condition is necessary:

The general equation of second degree is

$$ax^2+2hxy+by^2+2gx+2fy+c = 0 \quad \cdots (1)$$

Let (1) represents a pair of straight lines.

Then the left side of (1) can be resolved into two linear factors.

Arranging the equation (1) as a quadratic in x, we have

$ax^2 + 2(hy+g)x + by^2 + 2fy + c = 0$

So, $x = \dfrac{-(hy+g) \pm \sqrt{(hy+g)^2 - a(by^2+2fy+c)}}{a}$, if $a \neq 0$

Hence left side of (1) can be resolved into two factors

if $(hy+g)^2 - a(by^2+2fy+c)$

$= y^2(h^2-ab) + 2(gh-af)y + (g^2-ac)$

be a perfect square.

This is a quadratic in y and is a perfect square if its discriminant be zero, that is, if

$$4(gh-af)^2 - 4(h^2-ab)(g^2-ac) = 0$$

or, $g^2h^2 + a^2f^2 - 2afgh - g^2h^2 + abg^2 + ach^2 - a^2bc = 0$

or, $a(af^2 + bg^2 + ch^2 - abc - 2fgh) = 0$

If $a \neq 0$, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ — (2)

Since equation (1) is a second degree equation, at least one of a and b be non-zero. If $a = 0$, then $b \neq 0$.

Now arranging the equation (1) as a polynomial in we can prove (2), when $a = 0$.

The condition is sufficient : Let condition (2) holds. Then it is always possible to find x_1 and y_1 to satisfy

$$ax_1 + hy_1 + g = 0$$
$$hx_1 + by_1 + f = 0$$
and $$gx_1 + fy_1 + c = 0$$

as
$$ax + hy + g = 0$$
$$hx + by + f = 0$$
and $$gx + fy + c = 0$$

is a homogeneous system of linear equation whose coefficient matrix is $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

and $|A| = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ as (2) holds

Now $x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) + \cancel{x}\, gx_1 + fy_1 + c = 0$

gives $ax_1^2 + by_1^2 + 2hx_1y_1 + 2gx_1 + 2fy_1 + c = 0$. Shifting the origin to the point (x_1, y_1), the equation (1) reduces to the form $aX^2 + 2hXY + bY^2 = 0$

which represents the pair of straight lines

$$bY = \left(-h \pm \sqrt{h^2 - ab}\right)X$$

which are real and distinct, coincident or imaginary according as $h^2 > ab$, $h^2 = ab$ or $h^2 < ab$

Thus the necessary and sufficient condition that (1) will represent a pair of straight lines is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Corollary: If the equation (1) represents two straight lines, then the point of ~~interest~~ intersection (α, β) of the pair of straight lines represented by (1) is given by the equations $a\alpha + h\beta + g = 0$

and $h\alpha + b\beta + f = 0$

So, we get $\alpha = \dfrac{bf - ag}{ab - h^2}$; $\beta = \dfrac{gh - af}{ab - h^2}$ if $ab - h^2 \neq 0$

and the lines will be parallel if $ab - h^2 = 0$.

3.6. A Theorem

If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, Then $ax^2+2hxy+by^2=0$ represents a pair of straight lines through the origin parallel to the first pair

Proof: Let $ax^2+2hxy+by^2+2gx+2fy+c=0$ ... (1)

represents a pair of straight lines, then the expression in the left hand side can be resolved into two linear factors and (1) can be written as

$(lx+my+n)(l_1x+m_1y+n_1)=0$, where $ll_1=a$, $mm_1=b$,

$nn_1=c$ $lm_1+l_1m=2h$, $ln_1+l_1n=2g$, $mn_1+m_1n=2f$.

The equations of the straight lines parallel to these lines through the origin are $lx+my=0$ and $l_1x+m_1y=0$.

The combined equation of the two straight lines is

$$(lx+my)(l_1x+m_1y)=0$$

or, $$ll_1x^2+(lm_1+l_1m)xy+mm_1y^2=0$$

or, $$ax^2+2hxy+by^2=0$$

Corollary 1. If ϕ be the angle between the straight lines represented by the equation (1), then it will have same value as the angle between the two lines parallel to them passing through the origin. So,

$$\tan\phi=\frac{2\sqrt{h^2-ab}}{a+b}$$

Corollary 2 If the two straight lines represented by (1) are perpendicular, then the lines parallel to them and passing through the origin are also perpendicular. So, $a+b=0$