Corollary 3 If the straight lines represented by (1) be parallel, let these be $lx+my+n=0$ and $lx+my+n'=0$ and then (1) can be written as

$$(lx+my+n)(lx+my+n')=0$$

So, $l^2=a$, $m^2=b$, $lm=h$

$$l(n+n')=2g \quad m(n+n')=2f \quad nn'=c$$

Hence $h^2=l^2m^2=ab$, ~~given~~ giving $\frac{a}{h}=\frac{h}{b}$

Again $2bg=bl(n+n')=lm^2(n+n')$

and $2hf=hm(n+n')=lm^2(n+n')$

So, $2bg=2hf$ or, $bg=hf$

or, $\frac{h}{b}=\frac{g}{f}$

Thus, for parallelism of two straight lines (1), we have

$$\frac{a}{h}=\frac{h}{b}=\frac{g}{f}.$$

Corollary 4 The equation of the pair of ~~straight line~~ bisectors of the angles between the two straight lines represented by (1) is

$$\frac{(x-\alpha)^2-(y-\beta)^2}{a-b}=\frac{(x-\alpha)(y-\beta)}{h}$$

where (α,β) is the point of intersection of the two lines (1). It follows simply by shifting the origin to (α,β) and then finding the pair of bisectors.

3.7 To find the equation of the pair of straight lines joining the origin to the points where the straight line $lx + my = n$ --- (1) meets the curve

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ --- (2)

Let the line (1) meets the curve at P and Q. We are to find the equation of the pair of straight lines OP and OQ

We make the equation (2) homogeneous of degree two in x, y with the help of (1) and get

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)\left(\frac{lx+my}{n}\right) + c\left(\frac{lx+my}{n}\right)^2 = 0 \quad \text{---} (3)$$

The equation (3) is homogeneous of degree ~~a~~ two, so it represents a pair of straight lines through the origin. Moreover the equation (3) is satisfied by the co-ordinates of P and Q, because the co-ordinates of P and Q satisfy both equation (1) and (2) and hence (3).

Therefore (3) is the required equation of the pair of straight lines OP and OQ.

Worked Examples

1. Show that the equation $x^2 + 3xy + 2y^2 = 0$ represents a pair of straight lines. Find these straight lines and angle between them.

Solution: The given equation is a homogeneous equation of second degree. So, it represents a pair of straight lines through the origin.

The given equation may be written as

$$(x+y)(x+2y)=0$$

Hence the two straight lines are $x+y=0$ and $x+2y=0$

The angle between them is

$$\tan^{-1}\frac{2\sqrt{\left(\frac{3}{2}\right)^2-1.2}}{1+2} = \tan^{-1}\frac{1}{3}$$

2. Show that the equation of the pair of straight lines through the origin perpendicular to the pair of straight lines $ax^2+2hxy+by^2=0$ is

$$bx^2-2hxy+ay^2=0.$$

Solution: Let $y-m_1x=0$ and $y-m_2x=0$ be the straight lines represented by the equation $ax^2+2hxy+by^2=0$, so that $m_1+m_2=-\frac{2h}{b}$ and $m_1m_2=\frac{a}{b}$ ... (1)

The straight lines perpendicular to $y-m_1x=0$ and $y-m_2x=0$ passing through the origin are

$y+\frac{1}{m_1}x=0$ and $y+\frac{1}{m_2}x_2=0$ respectively.

Hence the equation of the required pair of straight lines is $\left(y+\frac{1}{m_1}x\right)\left(y+\frac{1}{m_2}x\right)=0$

or, $(m_1y+x)(m_2y+x)=0$

or, $m_1m_2y^2+(m_1+m_2)xy+x^2=0$

or, $bx^2-2hxy+ay^2=0$ → by (1)

Ex 3. Prove that the equation to the straight lines through the origin each of which makes an angle α with the straight line $y = x$ is

$$x^2 - 2xy\sec\alpha + y^2 = 0$$

Solution: The slope of the straight line $y = x$ is $1 = \tan\frac{\pi}{4}$. So the straight line makes an angle $\frac{\pi}{4}$ with the x-axis.

Hence the required straight lines make angles $\left(\frac{\pi}{4} - \alpha\right)$ and $\left(\frac{\pi}{4} + \alpha\right)$ with the x-axis.

As they pass through the origin, their equations are

$$y = x\tan\left(\frac{\pi}{4} - \alpha\right) \quad \text{and} \quad y = x\tan\left(\frac{\pi}{4} + \alpha\right)$$

that is, $y - \frac{1-\tan\alpha}{1+\tan\alpha}x = 0$ and $y - \frac{1+\tan\alpha}{1-\tan\alpha}x = 0$

Therefore the required equation of the pair of straight lines is

$$\left(y - \frac{1-\tan\alpha}{1+\tan\alpha}x\right)\left(y - \frac{1+\tan\alpha}{1-\tan\alpha}x\right) = 0$$

or, $y^2 - \left(\frac{1-\tan\alpha}{1+\tan\alpha} + \frac{1+\tan\alpha}{1-\tan\alpha}\right)xy + x^2 = 0$

or, $y^2 - \frac{2(1+\tan^2\alpha)}{1-\tan^2\alpha}xy + x^2 = 0$

or, $x^2 - \frac{2xy}{\cos 2\alpha} + y^2 = 0$

or, $x^2 - 2xy\sec 2\alpha + y^2 = 0$

4. If the pair of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each