2. System of linear equations:

A linear equation in n unknowns $x_1, x_2, \ldots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b \quad \cdots (1)$$

If in equation (1), $b = 0$, it is called a homogeneous equation. Otherwise, it is called a non-homogeneous equation.

Example: $2x_1 + 3x_2 + 4x_3 = 5$ is a non-homogeneous linear equation.

$3x_1 + 4x_2 + 7x_3 = 0$ is a homogeneous linear equation.

A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ can be written as

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \cdots (2)$$

If $b_1 = b_2 = \cdots = b_m = 0$, the system (2) is a homogeneous system of linear equations. Otherwise, it is said to be a system of non-homogenous linear equations.

Examples: 1.

$$\left.\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 0 \\ 5x_1 + 7x_2 + 3x_3 &= 0 \end{aligned}\right\} \cdots (3)$$

(3) is a homogeneous system of linear equations.

2.

$$\left.\begin{aligned} 3x_1 + 4x_2 + 5x_3 &= 0 \\ 4x_1 + 3x_2 + 7x_3 &= 5 \end{aligned}\right\} \cdots (4)$$

(4) is a ~~system of~~ ~~non-hom~~ system of non-homogenous system of linear equations.

3.

$$\left.\begin{aligned} 2x_1 + 5x_2 + 3x_3 &= 7 \\ 4x_1 + 5x_2 + 7x_3 &= 5 \end{aligned}\right\} \cdots (5)$$

(5) is a non-homogeneous system of linear equations.

If at least one set of values of $x_1, x_2, \ldots, x_n$ can be found which satisfy the system of equations (2), then the system of equations (2) is said to be consistent. If no such set is possible, (2) is said to be inconsistent.

In the former case the values of $x_1, x_2, \dots, x_n$ for which the system (2) ~~...~~ is satisfied are said to constitute the solutions of (2).

Consider the system

$$\left.\begin{aligned} 2x + 5y &= 9 \\ x - y &= 1 \end{aligned}\right\} \quad \dots (6)$$

The system (6) is consistent as it has unique solution $x = 2,\ y = 1$.

For the system,

$$\left.\begin{aligned} x + 2y &= 7 \\ 4x + 8y &= 28 \end{aligned}\right\} \quad \dots (7)$$

$x = 7 - 2c,\ y = c$ is always a solution for any arbitrary value of c. So, (7) has infinite number of solutions. So, it is a consistent system.

Consider the system

$$\left.\begin{aligned} 2x + 3y &= 5 \\ 4x + 6y &= -8 \end{aligned}\right\} \quad \dots (8)$$

System (8) is inconsistent as it has no solution.

In our H.S. course, we have learned Cramer's rule and matrix method for the solution of a system of non-homogeneous linear equation in three unknowns for unique solution:

The system (2) can be written in matrix form as $Ax = b$ where A is an $m \times n$ matrix given by

$$A = [a_{ij}]_{m\times n} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

A is said to be the coefficent matrix can be written in full form as
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

~~The $(m+1)\times n$ matrix augmented~~ ~~obtained by~~

The $m\times(n+1)$ matrix formed by augmenting or adding to a column which is b is said to be the augmented matrix and denoted by $[A, b]$. So,

$$[A, b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \\ a_{m1} & a_{m2} & & a_{mn} & b_m \end{bmatrix}$$

Consistency of a system of linear equations

$$Ax = b \quad \text{where } A = [a_{ij}]_{m\times n}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \cdots (9)$$

we state a theorem without proof

Theorem 1. The system (9) is consistent if and only if rank of A = rank of $[A, b]$

So, ~~$Ax=b$~~ is ~~consistent if~~ if rank of A = rank of $[A, b]$ then the system $Ax = b$ is consistent and conversely, if the system is consistent, rank of A = rank of $[A, b]$

Number of solutions of a non-homogeneous system $Ax = b$.

(i) If $r(A) \neq r([A, b])$ then the system is inconsistent and has no solution. ($r(A)$ = rank of A)

If $r(A) = r([A, b])$ = number of unknowns, then the system is consistent and they have a unique solution

If $r(A) = r([A, b]) \neq$ number of unknowns, then the equations are consistent but they have an infinite number of solutions.

So, for consistency of the system of equations $Ax = b$, $r(A) = r([A, b])$. Let $r(A) = r([A, b]) = r$. If $r < m$; then the elementary row transformations will eliminate $(m-r)$ of the equations and the m given equations will be replaced by an equivalent system of r equations. The ~~syst~~ equivalent system means, the original system and the reduced system after use of elementary row operations have the ~~set~~ same set of solutions. From these r equations we can express the values of r of the unknowns in terms of the remaining $(n-r)$ unknowns and we can give arbitrary values to these unknowns.

The following cases may arise:

(i) If $r = n$, that is, if $n - r = 0$, then no variable will be given an arbitrary value and the solution will be unique.

(ii) If $r < n$, that is, $n - r > 0$, then arbitrary values will be given to $(n-r)$ variables and thus there will be infinite number of solutions.

This is also the case, if $m < n$, that is, if $r \leq m < n$ and $n - r > 0$

In case $m > n$, the system is consistent if $r \leq n$. The system has a unique solution for $r = n$ and has an infinite number of solutions for $r < n$.