straight lines given by the equation

$$2x^2-5xy+3y^2-2x+3y=0$$

Solution: The given equation is written as

$$2x^2 - x\,(5y+2) + (3y^2+3y) = 0$$

So, $$x = \frac{5y+2 \pm \sqrt{(5y+2)^2-8(3y^2+3y)}}{4}$$

or, $$4x-5y-2 = \pm\sqrt{y^2-4y+4} = \pm(y-2)$$

Thus the equations of the straight lines are

$$4x-6y=0 \quad \text{and} \quad 4x-4y-4=0$$

that is, $2x-3y=0$ and $x-y-1=0$

Solving, we get $x=3$ and $y=2$

So, the point of intersection of the pair of straight lines is $(3,2)$. So, the required distance $= \sqrt{3^2+2^2} = \sqrt{13}$

10. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two lines equidistant from the origin, show that $f^4-g^4=c(bf^2-ag^2)$

Solution: Let $ax^2+2hxy+by^2+2gx+2fy+c$
$$=(l_1x+m_1y+n_1)(l_2x+m_2y+n_2)$$

Comparing the coefficients, $l_1l_2=a$, $m_1m_2=b$, $l_1m_2+l_2m_1=2h$, $l_1n_2+l_2n_1=2g$, $m_1n_2+m_2n_1=2f$ $n_1n_2=c$

The lines are $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$

Since the ~~2~~ lines are equidistant from the origin,

$$\left|\frac{n_1}{\sqrt{l_1^2+m_1^2}}\right| = \left|\frac{n_2}{\sqrt{l_2^2+m_2^2}}\right|$$

or, $n_1^2\left(l_2^2+m_2^2\right) = n_2^2\left(l_1^2+m_1^2\right)$

or, $n_1^2 l_2^2 - n_2^2 l_1^2 = n_2^2 m_1^2 - n_1^2 m_2^2$

or, $(n_1 l_2 + n_2 l_1)(n_1 l_2 - n_2 l_1) = (n_2 m_1 + n_1 m_2)(n_2 m_1 - n_1 m_2)$

or, $2g\,(n_1 l_2 - n_2 l_1) = 2f\,(n_2 m_1 - n_1 m_2)$

or, $g^2\left\{(n_1 l_2 + n_2 l_1)^2 - 4n_1 n_2 l_1 l_2\right\} = f^2\left\{(n_2 m_1 + n_1 m_2)^2 - 4 m_1 m_2 n_1 n_2\right\}$

or, $g^2\left(4g^2 - 4ca\right) = f^2\left(4f^2 - 4bc\right)$

or, $f^4 - g^4 = c\left(bf^2 - ag^2\right)$

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4. Equations of pair of tangents from an external points, Chord of contact, poles and polars in case of General conic. Particular cases for Parabola, Ellipse, Circle, Hyperbola.

4.1 ~~A~~ <u>Pair of tangents</u>

(a) A pair of tangents can be drawn to a conic from an external point, i.e., a point not lying on the conic.

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ~~(1)~~ $\cdots (1)$

be the equation of the conic and (x_1, y_1) be an external point. Let a line through the point (x_1, y_1) touch the conic at (x_2, y_2). The equation of the tangent at (x_2, y_2) is

$$a x x_2 + h(x y_2 + y x_2) + b y y_2 + g(x + x_2) + f(y + y_2) + c = 0$$

(because the equation of the tangent at (x_1, y_1) of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is
$a x x_1 + h(x y_1 + x_1 y) + g(x + x_1) + f(y + y_1) + c = 0$)

As it passes through (x_1, y_1), we have

$$a x_1 x_2 + h(x_1 y_2 + x_2 y_1) + b y_1 y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0 \quad \text{---}(2)$$

Again (x_2, y_2) is on the conic (1), we have

$$a x_2^2 + 2h x_2 y_2 + b y_2^2 + 2g x_2 + 2f y_2 + c = 0 \quad \text{---}(3)$$

From (2) and (3) generally two values of x_2 and y_2 are obtained. Thus there will two points of contact of tangents from (x_1, y_1) but they may not be real in all cases.

(b) Equation of the pair of tangents

The equation of a line through the point (x_1, y_1) can be written as $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = r \text{ (say)}$ ---(4)

From (4) $x = lr + x_1$, $y = mr + y_1$. Putting these values of x and y in the equation (1), we have

$$(al^2+2hlm+bm^2)r^2+2\{(ax_1+hy_1+g)l+(hx_1+by_1+f)m\}r$$
$$+ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0 \quad \cdots (5)$$

It is a quadratic equation in r. If the line (4) touches the conic (1), the equation (5) must have equal roots. By the condition of equal roots

$$\{(ax_1+hy_1+g)l+(hx_1+by_1+f)m\}^2$$
$$=(al^2+2hlm+bm^2)(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)$$

Eliminating l and m by (4)

$$\{(ax_1+hy_1+g)(x-x_1)+(hx_1+by_1+g)(y-y_1)\}^2$$
$$=\{a(x-x_1)^2+2h(x-x_1)(y-y_1)+b(y-y_1)^2\}\times$$
$$(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c).$$
$$\cdots (6)$$

If we write,

$$S=ax^2+2hxy+by^2+2gx+2fy+c$$
$$S_1=ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c$$

and $T=axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c$,

then the equation (6) can be written as

$$(T-S_1)^2=(S+S_1-2T)S_1 \quad \text{or,} \quad SS_1=T^2 \cdots (7)$$

It is the required equation.

<u>Director Circle</u> (Locus of the points of intersection of pair of perpendicular tangents)