Definition: If the locus of the points of intersection of pair of ~~tangents~~ perpendicular tangents to a conic is a circle, then the circle is called the director circle of the conic.

(i) Circle: $x^2+y^2=a^2$.

The equation of the pair of tangents from (x_1, y_1) to the circle is $(x^2+y^2-a^2)(x_1^2+y_1^2-a^2) = (xx_1+yy_1-a^2)^2$

If these are at right angles, then

the coefficient of x^2 + coefficient of $y^2 = 0$

i.e., $(x_1^2+y_1^2-a^2-x_1^2)+(x_1^2+y_1^2-a^2-y_1^2)=0$

or, $x_1^2+y_1^2=2a^2$

Hence the locus of (x_1, y_1), i.e., the equation of the director circle is $x^2+y^2=2a^2$

(ii) Parabola: $y^2=4ax$

The equation of the pair of tangents from (x_1, y_1) to the parabola is $(y^2-4ax)(y_1^2-4ax_1) = (yy_1-2ax-2ax_1)^2$

If these lines are perpendicular, then

$4a^2+y_1^2-y_1^2+4ax_1=0$ or, $x_1+a=0$

Hence the locus of (x_1, y_1) is $x+a=0$. It is the straight line representing the directrix of the parabola. Thus the points of intersection of perpendicular tangents to a parabola lie on the directrix.

(iii) Ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The equation of the pair of tangents at (x_1, y_1) to the ellipse is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

If these lines are perpendicular to each other

$$\frac{1}{a^2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)-\frac{x_1^2}{a^4}+\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)-\frac{y_1^2}{b^4}=0$$

or, $x_1^2+y_1^2=a^2+b^2$

Hence the director circle of the ellipse is $x^2+y^2=a^2+b^2$

(iv) Hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

The equation of the pair of tangents from (x_1, y_1) to the hyperbola is

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2$$

For the pair of tangents to be perpendicular

$$\frac{1}{a^2}\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)-\frac{x_1^2}{a^4}-\frac{1}{b^2}\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)-\frac{y_1^2}{b^4}=0$$

or, $x_1^2+y_1^2=a^2-b^2$.

Hence the equation of the director circle is

$$x^2+y^2=a^2-b^2.$$

4.2 Chord of contact

Definition: It is a chord joining the points of contact of tangents to a conic from a given point not lying on the conic.

Let $ax^2+2hxy+by^2+2gx+2fy+c=0$ ... (1)

be the equation of the conic and (x_1, y_1) be the point from which tangents are drawn to the conic.

Let (x_2, y_2) and (x_3, y_3) be the points of contact. The equations of tangents at (x_2, y_2) and (x_3, y_3) are

$$axx_2 + h(xy_2+yx_2) + byy_2 + g(x+x_2) + f(y+y_2) + c = 0 \quad \cdots (2)$$

and $$axx_3 + h(xy_3+yx_3) + byy_3 + g(x+x_3) + f(y+y_3) + c = 0 \quad \cdots (3)$$

Since these two tangents pass through the point (x_1, y_1)

$$ax_1x_2 + h(x_1y_2 + y_1x_2) + by_1y_2 + g(x_1+x_2) + f(y_1+y_2) + c = 0$$

and $$ax_1x_3 + h(x_1y_3 + y_1x_3) + by_1y_3 + g(x_1+x_3) + f(y_1+y_3) + c = 0$$

The above two conditions suggest that the line

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x+x_1) + f(y+y_1) + c = 0 \quad \cdots (4)$$

passes through (x_2, y_2) and (x_3, y_3). Hence it is the equation of the chord of contact of tangents through (x_1, y_1).

Note: Here the equation (4) is identical with the equation of tangent at (x_1, y_1) on the ~~coot~~ conic. But here (x_1, y_1) does not lie on the conic.

Example 1 Find the point of intersection of the tangents at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ to the parabola $y^2 = 4ax$

Solution Let (x_1, y_1) be the point of intersection of the two tangents. The equation of the chord of contact of the tangents from (x_1, y_1) is

$$yy_1 = 2a(x + x_1)$$

Since it passes through the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$, we have

$$2at_1 y_1 = 2a(at_1^2 + x_1)$$

or, $\quad t_1 y_1 = x_1 + at_1^2 \quad \cdots (1)$

Similarly, $\quad t_2 y_2 = x_1 + at_2^2 \quad \cdots (2)$

Solving (1) and (2), we have $x_1 = at_1t_2$, and $y_1 = a(t_1 + t_2)$

Hence the point of intersection is $(at_1t_2, a(t_1+t_2))$

4.3 Pole and Polar.

Definition: The polar of a point with respect to a conic is the locus of the points of intersection of tangents at the extremities of the