Therefore $\frac{x'}{l} = \frac{y'}{m} = \frac{-a^2}{n}$. So, $x' = -\frac{a^2 l}{n}$, $y' = -\frac{a^2 m}{n}$

Then the pole is at $\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right)$

(b) To find the pole of the straight line $lx+my+n=0$ with respect to the parabola $y^2 = 4ax$.

Let the pole of the given straight line be (x', y').

Now the polar of (x', y') with respect to the parabola $y^2 = 4ax$ is $yy' = 2a(x+x')$.

This is then identical with $lx+my+n=0$

Therefore $\frac{2a}{l} = \frac{-y'}{m} = \frac{2ax'}{n}$. So $x' = \frac{n}{l}$, $y' = -\frac{2am}{l}$

Then the pole is at $\left(\frac{n}{l}, -\frac{2am}{l}\right)$.

(c) To find the pole of the straight line $lx+my+n=0$ with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let the pole of the given straight line be (x', y')

Now the polar of (x', y') with respect to ellipse is $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$.

This is then identical with $lx+my+n=0$

Therefore $\frac{x'/a^2}{l} = \frac{y'/b^2}{m} = \frac{-1}{n}$. So,

$x' = -\frac{a^2 l}{n}$, $y' = -\frac{b^2 m}{n}$

Thus the pole is at $\left(-\frac{a^2 l}{n}, -\frac{b^2 m}{n}\right)$.

Corollary: Similarly, the pole of the straight line $lx+my+n=0$ with respect to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ is at } \left(-\frac{a^2 l}{n}, \frac{b^2 m}{n}\right).$$

Worked Examples

1. (a). Find the equation of the polar of the point $(2,3)$ with respect to the circle $x^2+y^2-2x-4y+1=0$

(b) Find the coordinates of the pole of the straight line $x-8y=12$ with respect to the conic $\frac{x^2}{36}-\frac{y^2}{9}=1$

Solution: (a). The equation of the polar of the point (x_1, y_1) with respect to the given circle is

$$xx_1+yy_1-(x+x_1)-2(y+y_1)+1=0$$

Here $x_1=2$, $y_1=3$. Hence the required equation of the polar is $2x+3y-(x+2)-2(y+3)+1=0$

or, $x+y=7$

(b) Let the coordinates of the pole be (x', y'). Equation of the polar of the point (x', y') with respect to the conic $\frac{x^2}{36}-\frac{y^2}{9}=1$ is

$$\frac{xx'}{36}-\frac{yy'}{9}=1 \quad \cdots (1)$$

If
$$x-8y=12 \quad \cdots (2)$$

be the polar of the point (x', y') with respect to the conic $\frac{x^2}{36}-\frac{y^2}{9}=1$, then (1) and (2) are identical. Hence

$$\frac{x'/36}{1} = \frac{y'/9}{8} = \frac{1}{12}$$

So, $x' = 3$, $y' = 6$

So, The coordinates of the required pole is $(3, 6)$

2. The polar of the point P with respect to the circle $x^2+y^2=a^2$ touches the circle $4x^2+4y^2=a^2$. Show that the locus of P is the circle $x^2+y^2=4a^2$

Solution: Let P be the point (α, β). Then the polar of P with respect to the circle $x^2+y^2=a^2$ is $\alpha x+\beta y=a^2$. Since it touches the circle $4x^2+4y^2=a^2$, that is, $x^2+y^2=\left(\frac{1}{2}a\right)^2$, its perpendicular distance from the centre $(0,0)$ is equal to the radius $\frac{1}{2}a$. Therefore

$$\frac{a^2}{\sqrt{\alpha^2+\beta^2}} = \pm\frac{1}{2}a$$

or, $\alpha^2+\beta^2=4a^2$. Hence the locus of $P(\alpha,\beta)$ is: $x^2+y^2=4a^2$

3. If the pole of the straight line with respect to the circle $x^2+y^2=a^2$ lies on $x^2+y^2=k^2a^2$, then prove that the straight line will touch the ~~circle~~

circle $x^2+y^2 = a^2/k^2$

Solution: Let (x_1, y_1) be a point on the circle

$x^2+y^2=k^2a^2$, so that

$$x_1^2+y_1^2 = k^2a^2 \quad \cdots \quad (1)$$

Its polar with respect to the circle $x^2+y^2 = a^2$ is

$$xx_1+yy_1 = a^2$$

It will touch the circle $x^2+y^2 = a^2/k^2$, if its perpendicular distance from the centre $(0,0)$ be equal to the radius a/k,

that is, if $\dfrac{a^2}{\sqrt{x_1^2+y_1^2}} = \pm a/k$

that is, if $x_1^2+y_1^2 = a^2k^2$, which is true by (1)

4. Prove that the locus of the poles of the normal chords of the parabola $y^2=4ax$ is the curve

$$y^2(x+2a)+4a^3=0$$

Solution: The equation of the normal to the parabola $y^2=4ax$ at the point $(at^2, 2at)$ is

$$y+tx = 2at+at^3 \quad \cdots \quad (1)$$

Let the pole of this straight line with respect to the parabola be (α, β). The equation of the polar of (α, β) with respect to the parabola