Homogeneous system of linear equations.

In the case of homogeneous system of linear equations, $b_1 = b_2 = \cdots = b_m = 0$ in (2). So, the system becomes

$$Ax = 0 \quad \ldots (9) \quad \text{where } A = [a_{ij}]_{m \times n} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and $O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

$x_1 = x_2 = \cdots = x_n = 0$ is always a solution of the homogeneous system (9). So, homogeneous system is always a consistent system. Actually, $r(A) = r[A, O]$, so, the system is consistent. $x_1 = x_2 = \cdots = x_n = 0$ is called a trivial solution of (9). So, for any non-trivial solution of (9) at least one of $x_1, \ldots x_n$ is non-zero, if such a solution exists.

If the coefficient matrix A is a square matrix and non-singular then the system (9) has only the trivial solution as $Ax = 0 \Rightarrow A^{-1}(Ax) = A^{-1}O = O$ $\Rightarrow x = O$. So, $x_1 = x_2 = \cdots = x_n = 0$

When A is a square matrix and A is singular, that is, $|A| = 0$ ($|A|$ is determinant of A, also denoted by det A) we give an example to show that the system (9) will have a non-zero solution. In fact, the system has infinite number of solutions.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here $|A| = 0$. Applying row operations $R_2 - 2R_1$ and $R_3 - 4R_1$

the system is equivalent to

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In the system of equivalent equations, the last two equations $-x_2 - 2x_3 = 0$ and $-3x_2 - 6x_3 = 0$ are the same and it $x_2 = -2x_3$. The first equation is $x_1 + 2x_2 + 3x_3 = 0$, which on substitution of $x_2 = -2x_3$, reduces to $x_1 - x_3 = 0$ or, $x_1 = x_3$

In this example, A is singular and its rank is 2 which is less than the order of the matrix. Here we get x_1 and x_2 in terms of x_3, which can be chosen arbitrarily. Hence there will be an infinite number of solutions.

Again, if we consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ in the System $AX = O$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

then $|A| = 0$, so that A is singular

using row operations $R_2 - 2R_1$ and $R_3 - 3R_1$, the system is equivalent to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, there is only one equation

$$x_1 + x_2 + x_3 = 0$$

So, $x_1 = -x_2 - x_3$

Hence two variables can be chosen arbitrarily and the system will have an infinite number of solutions

Thus a system of homogeneous linear equations may ~~possesses~~ possess only the trivial solution or it may possess infinitely many non-trivial solutions besides the trivial solution.

Hence ~~of~~ we see that if $r(A)=r$ where A is an $m\times n$ matrix and $r=n\,(\le m)$, then there is a unique solution which is the trivial solution.

If $r<n\,(\le m)$, then $n-r$ variables can be selected and assigned ~~arbitrarily~~ arbitrary values and hence there is an infinite number of solutions.

If $m<n$, $r\le m<n$, $(n-r)$ variables can be selected and assigned arbitrary values and hence there is an infinite number of solutions. Here the number of equations is less than the number of variables.

<u>Worked Examples</u>: 1. Show the system of equations

$$\begin{aligned} 2y+4z+5&=0\\ 8x-y+4z&=12\\ 16x-y+10z&=1 \end{aligned}$$

is inconsistent.

Solution: Here the coefficient matrix $A=\begin{bmatrix}0&2&4\\8&-1&4\\16&-1&10\end{bmatrix}$

and $[A,b]=\begin{bmatrix}0&2&4&-5\\8&-1&4&12\\16&-1&10&1\end{bmatrix}$ where $b=\begin{bmatrix}-5\\12\\1\end{bmatrix}$ and $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$

~~Homogeneous system of linear equations.~~

~~In the case of homogeneous system of linear equations, $b_1 = b_2 = \cdots = b_m = 0$ in (2), so the system becomes~~

$$\cancel{Ax = O \quad \text{where } A = [a_{ij}]_{m\times n},\ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}$$

$$\cancel{\text{and } O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

Here $|A| = -2(80-64) + 4(-8+16) = -32+32 = 0$

So, rank of A is not 3. Consider a minor $\begin{vmatrix} 0 & 2 \\ 8 & -1 \end{vmatrix}$

of order 2 of A. This is not zero.

Hence rank of A is 2

Now we consider the minor $\begin{vmatrix} 2 & 4 & -5 \\ -1 & 4 & 12 \\ -1 & 10 & 1 \end{vmatrix}$

of order of $[A, b]$. The value of the minor

is $2(4-120) - 4(-1+12) - 5(-10+4) = -246 \neq 0$

Hence rank of $[A, b]$ is 3

So, rank of $A \neq$ rank of $[A, b]$

So, the system is inconsistent.

2. Discuss the consistency and the solutions of the equations $x + ay + az = 1$, $ax + y + 2az = -4$, $ax - ay + 4z = 2$ for different values of a.

Solution: Here the coefficient matrix

$$A = \begin{bmatrix} 1 & a & a \\ a & 1 & 2a \\ a & -a & 4 \end{bmatrix}$$ and the augmented